# Lecture 27 (2-11), Nov 21, 2022

## Quantum Tunneling

- In classical mechanics, when a particle reaches a potential barrier greater than its energy, it will stop and turn back
- In the quantum case, because the wavefunction doesn't decay instantaneously, the particle have a finite probability to pass through a potential barrier
- This is known as *quantum tunneling*
- Conversely, in quantum mechanics even when the particle has enough energy to overcome a barrier, there is a finite probability that the particle is reflected instead

## Potential Step

- Consider potential $V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases}$
- Strategy: break up the potential into two regions, separately solve TISE, then match boundary conditions
  - In this case we have continuity of $\psi(x)$ as well as $\psi'(x)$
  - A discontinuity in $V(x)$ leads to a discontinuity of $\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}$, but if $V(x)$ is finite, then $\frac{\mathrm{d}\psi}{\mathrm{d}x}$ will still be continuous
    * This is why we need to consider $\psi'$ in this case, but not in the case of the infinite square well, because here $V(x)$ is finite but it's not in the infinite square well
- In the first region $x < 0 \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{2mE}{\hbar^2}\psi = 0 \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\psi = 0 \implies \psi_I(x) = Ae^{ikx} + Be^{-ikx}$
- In the second region $x > 0 \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{2m}{\hbar^2}(E - V_0)\psi = 0$
  - Define $\alpha^2 = \frac{2m}{\hbar^2}(V_0 - E)\psi \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \alpha^2\psi = 0$
    * Note this is because we're considering when $E < V_0$
  - This gives $\psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x}$
- We know $C = 0$ otherwise $\psi_{II}$ is not square integrable
- Matching initial conditions:
  - $\psi_I(0) = \psi_{II}(0) \implies A + B = D$
  - $\psi_I'(0) = \psi_{II}'(0) \implies Aik - Bik = -\alpha D$
- For now we're interested in the ratios $\frac{B}{A}$ and $\frac{D}{A}$
  - $1 + \frac{B}{A} = \frac{D}{A}$
  - $1 - \frac{B}{A} = -\frac{\alpha}{ik}\frac{D}{A}$
  - $2 = \left(1 - \frac{\alpha}{ik}\right)\frac{D}{A}$
  - $\left(1 + \frac{\alpha}{ik}\right) + \frac{B}{A}\left(1 - \frac{\alpha}{ik}\right) = 0$
  - $\frac{B}{A} = \frac{1 - \frac{i\alpha}{k}}{1 + \frac{i\alpha}{k}}$
  - $\frac{D}{A} = \frac{2}{1 + \frac{i\alpha}{k}}$
  - Note $\frac{\alpha}{k} = \sqrt{\frac{V_0 - E}{E}}$, so we can determine these by the ratio of the energies
- Note physically $A$ is the wave coming in, $B$ is the reflected wave, and $D$ is the transmitted wave in the classically forbidden region
- Define the reflection coefficient $R = \frac{|B|^2}{|A|^2} = \left|\frac{B}{A}\right|^2 = \frac{1 + i\frac{\alpha}{k}}{1 - i\frac{\alpha}{k}}\frac{1 - i\frac{\alpha}{k}}{1 + i\frac{\alpha}{k}} = 1$

– This means we have total reflection; even though when the reflection happens there's a finite probability of the particle in the forbidden region, eventually everything is reflected in the end