Lecture 26 (2-10), Nov 17, 2022

Free Particle

- Consider a free particle, i.e. $V(x) = 0$ everywhere
- $-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi = E\psi$
- Let $k^2 = \frac{2mE}{\hbar^2}$ and $\omega = \frac{E}{\hbar} = \frac{\hbar k^2}{2m}$
  - With these $\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\psi = 0$
  - Therefore $\psi = Ae^{ikx} + Be^{-ikx}$
- With this we can write the full solution as $\Psi(x,t) = \psi(x)e^{-\frac{E}{\hbar}t} = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$
  - This is a superposition of two travelling waves, one in to the left and one to the right
  - We can just take the first part and let $\Psi(x,t) = Ae^{i(kx-\omega t)}$
- Normalize: $\int |\psi|^2 \,\mathrm{d}x = \int A^2 \,\mathrm{d}x = \infty$, but this is not normalizable!
- $e^{i(kx-\omega t)}$ is a plane wave and it cannot be a wavefunction
- However, even though a single plane wave is not normalizable, if we superimpose many of them, the resulting *wave packet* becomes normalizable
- We can write $\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k)e^{i(kx-\omega t)} \,\mathrm{d}k$
  - $\phi(k)$ is the analogue of c_n and k is the analogue of n, except that k is not quantized
- For time $t = 0$: (works for any other finite value of t)
  - $\phi(k)$ is the Fourier transform of $\Psi(x,0)$
  - $\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k)e^{ikx} \,\mathrm{d}x$
  - $\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x,0)e^{-ikx} \,\mathrm{d}x$

Heisenberg Uncertainty Principle

- Notice that as $\phi(k)$ becomes more concentrated in k, $\psi(x)$ becomes more spread out in x
- For a wavepacket, Δx and Δk are inversely proportional: $\Delta x\Delta k \sim 1$
- Since $p = \hbar k$, we have $\Delta x\Delta p \sim \hbar$
- Often written as $\Delta x\Delta p \geq \frac{\hbar}{2}$
  - The exact relation is unimportant, but this is the fundamental limit
- We can also have time-energy uncertainty principle $\Delta t\Delta E \geq \frac{\hbar}{2}$
  - However this is different since t is not exactly an observable
  - Note if $\Delta E = 0$ we are in a stationary state, which has an infinite lifetime