# Lecture 24 (2-8), Nov 14, 2022

## Infinite Square Well (1D)

- Bound state, localized in $\Delta x$
- $V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}$
  - The particle cannot be outside $(0, a)$ since the potential is infinite there
- Outside the well, $\psi = 0$ is the only thing that can satisfy the SE
- Inside the well: $\hat{H}\psi = E\psi \implies -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi = E\psi$
  - $\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{2mE}{\hbar^2}\psi = 0$
    - When $E < 0$: Let $\alpha^2 = -\frac{2mE}{\hbar^2} \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \alpha^2\psi = 0 \implies \psi(x) = Ae^{\alpha x} + Be^{-\alpha x}$
    - When $E > 0$: Let $k^2 = \frac{2mE}{\hbar^2} \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\psi = 0 \implies \psi(x) = A\sin(kx) + B\cos(kx)$
- Enforce continuity conditions: $\psi(x) = 0$ for $x < 0$ or $x > a$
  - $\psi(0) = 0 \implies B = 0$
  - $\psi(a) = 0 \implies A\sin(ka) = 0 \implies ka = n\pi \implies k = \frac{n\pi}{a}$
    - Assume $A$ is nonzero because that leads to a non-normalizable solution
  - This gives $E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2 n^2}{2ma^2}$
  - Note we're not considering continuity of the derivative here because our potential is infinite
- Normalization condition: $\int_{-\infty}^{\infty} |\psi|^2 \,\mathrm{d}x = 1 \implies A = \sqrt{\frac{2}{a}}$
- $\psi_n(x) = A\sin(k_n x) = A\sin\left(\frac{n\pi}{a}x\right) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$
- There are an infinite number of allowed solutions; $n = 1$ is the lowest energy state (ground state)
  - Notice it's not at zero energy! This lowest energy level is the *zero-point energy*
- The full solution is $\Psi_n(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{i\hbar\pi^2 n^2}{2ma^2}t}$
  - The most general solution is a sum of all of these: $\Psi(x,t) = \sum_{n=1}^{\infty} C_n\Psi_n(x,t)$
  - In general $\Psi(x,t)$ is not a stationary state

## Uncertainty Principle

- Physically, why can't we have $n = 0$?
- Physically $E = 0 \implies p = 0 \implies \Delta p = 0$, so there is no uncertainty in momentum; this would mean $\Delta x\Delta p = 0$, violating the uncertainty principle
- What about $n = 1$?
  - $\Delta x$ is at most $a$ since the particle is never outside the box
  - $\Delta p$ can be estimated as $2\sqrt{2mE} = 2\frac{\pi\hbar}{a}$
    - $p = \sqrt{2mE}$ is incorrect; energy is a vector and momentum is a scalar
      - In 1D this means we have $p = \pm\sqrt{2mE}$
  - $\Delta x\Delta p = 2\pi\hbar$