Lecture 23 (2-7), Nov 3, 2022

Interpretation of the Wave Function

- If we measure the particle to be at x at time t, where was the particle just before the measurement?
  - Was it there all the time? Is this unknown? Is this even a valid question?
- Copenhagen interpretation (Max Born)
  - The position is indeterminate until the measurement – it's not known
  - At the point of the measurement the wavefunction *collapses* to position x
  - The wavefunction before the measurement is a superposition of all the possible wavefunctions describing the particle at every possible point
  - This is opposed to the classical interpretation, which would say the particle would be at x even before the measurement
    * This would indicate the existence of hidden variables and the incompleteness of quantum mechanics
    * This was shown to be not true by Bell's inequality

Operators

- If we have $P(x,t) = |\Psi(x,t)|^2$ we can calculate $\langle x \rangle = \int x|\Psi(x,t)|^2 \, \mathrm{d}x = \int \Psi^* x \Psi \, \mathrm{d}x$
- In quantum mechanics observable quantities (e.g. position, momentum) are represented by *operators*
  - e.g. $\hat{x}$ is the position operator
- The expectation value of operator $\hat{O}$ is $\Psi^* \hat{O} \Psi \, \mathrm{d}x$
  - This is the value actually measured in an experiment
- The momentum operator $\hat{p}$, how do we define it?
  - Consider the time derivative of position $\dfrac{\mathrm{d}}{\mathrm{d}t}$
  - $\dfrac{\mathrm{d}}{\mathrm{d}t} \langle \hat{x} \rangle = -\dfrac{\partial i\hbar}{\partial m} \displaystyle\int \Psi^* \dfrac{\partial \Psi}{\partial x} \, \mathrm{d}x = \dfrac{1}{m} \displaystyle\int \Psi^* \left(-i\hbar \dfrac{\partial}{\partial x} \right) \Psi \, \mathrm{d}x$
  - This gives us the momentum operator $\hat{p} = i\hbar \dfrac{\partial}{\partial x}$
- Ehrenfest's Theorem: expectation values follow classical mechanics

Separation of Variables

- If we assume $\Psi(x,t) = \psi(x)\phi(t)$ we can plug this into the SE
  - This gives us $i\hbar \dfrac{1}{\phi} \dfrac{\mathrm{d}\phi}{\mathrm{d}t} = -\dfrac{\hbar^2}{2m} \dfrac{1}{\psi} \dfrac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V(x)$
  - The left hand side is a function of only time and the right hand side is a function of only position, so they must be constants
  - Let both equal E
- For the left hand side we get $i\hbar \dfrac{1}{\phi} \dfrac{\mathrm{d}\phi}{\mathrm{d}t} = E \implies \phi = e^{-\frac{iE}{\hbar} t}$
- For the right hand side we get $-\dfrac{\hbar^2}{2m} \dfrac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V\psi = E\psi$, which is the time-independent Schrödinger equation
  - Notice $-\dfrac{\hbar^2}{2m} \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ is equal to $\dfrac{\hat{p}^2}{2m}$
  - Define this as $\hat{T} = \dfrac{\hat{p}^2}{2m}$, the kinetic energy operator
  - Define the potential energy operator $\hat{V} = V(x)$
  - Define the Hamiltonian as $\hat{T} + \hat{V}$
    * The Hamiltonian is the total energy and gives us energy conservation
    * $\langle \hat{H} \rangle = \displaystyle\int \psi^* \hat{H} \psi \, \mathrm{d}x = \displaystyle\int \psi^* E \psi \, \mathrm{d}x = E \displaystyle\int |\psi|^2 \, \mathrm{d}x = E$

  - This gives us $\hat{H}\psi = E\psi$
- Notice the TISE is an eigenvalue equation
  - For a given $V(x)$ and boundary conditions, there is a set of ψs that satisfy the equation with corresponding energy eigenvalue E
  - The boundary conditions lead to quantization in these solutions, which means that E, the total energy, is quantized
- Note $|\Psi(x,t)|^2 = |\psi(x)|^2$, which is why these states are called *stationary states*

Summary

The Hamiltonian is defined as

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) = \frac{\hat{p}^2}{2m} + \hat{V}$$

and the Time-Independent Schrödinger Equation can be formulated as

$$\hat{H}\psi(x) = E\psi(x)$$

The full time-dependent solution is then

$$\Psi(x,t) = \phi(t)\psi(x) = e^{-i\frac{E}{\hbar}t}\psi(x)$$