Lecture 22 (2-6), Nov 1, 2022

Tonomura Experiment

- Electron double-slit experiment shows interference fringes characteristic of waves
- Electrons are sent one at a time, somehow still show a wavelike distribution – how do the electrons already know where to go?
- Therefore matter does behave like a wave – but what equation governs this wave?

Wave Equation for a Matter Wave

- For both light and matter waves $E = h\nu = \hbar\omega$ and $p = \dfrac{h}{\lambda} = \hbar k$
- However their dispersion relations are different:
  - For photons $E = pc \implies c = \dfrac{E}{p} = \dfrac{\omega}{k}$, so ω and k are linear in relationship
  - For matter $E = \dfrac{p^2}{2m} + V \implies \omega \propto k^2$, so ω is quadratic in k
- The classical wave equation, $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial x^2}$, would not work for matter waves
  - Take $y = A\sin(kx - \omega t) \implies \dfrac{\partial^2 y}{\partial x^2} = k^2 y, \dfrac{\partial^2 y}{\partial t^2} = \omega^2 y$
  - This can only be satisfied if the relationship between ω and k is linear
  - Matter waves cannot satisfy this due to their different dispersion relation
    - For a matter wave, we have to differentiate ω only once, but that means we have to match a sine and a cosine
    - This suggests we use a complex exponential: $\Psi(x, t) = Ae^{i(kx-\omega t)}$
      - $\dfrac{\partial^2 \Psi}{\partial x^2} = (ik)^2 Ae^{i(kx-\omega t)} = -k^2\Psi = -\left(\dfrac{p}{\hbar}\right)^2 \Psi$
      - $\dfrac{\partial \omega}{\partial x} = (-i\omega)Ae^{i(kx-\omega t)} = -i\omega\Psi = -i\dfrac{E}{\hbar}\Psi$
      - We can relate the two: $\dfrac{\partial^2 \Psi}{\partial x^2} = -\dfrac{p^2}{\hbar^2}\Psi = -\dfrac{2m(E+V)}{\hbar^2}\Psi = -\dfrac{2mE}{\hbar^2}\Psi + \dfrac{2mV}{\hbar^2} = -\dfrac{2m}{\hbar}\left(i\hbar\dfrac{\partial\Psi}{\partial t}\right) + \dfrac{2m}{\hbar^2}V\Psi$
- This leads us to the (time-dependent) Schrödinger equation: $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\dfrac{\partial\Psi}{\partial t}$
  - We can alternatively express this as $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi}{\partial x^2} + V\Psi = E\Psi$, which is the time-independent Schrödinger equation

Properties of the Matter Wavefunction

1. $|\Psi(x,t)|^2 = P(x,t)$, the probability distribution/density
   - The probability of finding a particle in the interval $[x, x + \mathrm{d}x]$ is $|\Psi(x)|^2\,\mathrm{d}x = P(x)\,\mathrm{d}x$
2. The wavefunction is square-integrable
   - Since we're interpreting $|\Psi(x,t)|^2$ as the probability density, then it must be true that $\displaystyle\int_{-\infty}^{\infty} |\Psi|^2\,\mathrm{d}x = 1$ (normalization condition)
   - We satisfy this by scaling Ψ by a number, but this requires that the integral is finite and nonzero
3. Two waves can be superimposed and their sum will still satisfy the wavefunction
4. Ψ must be continuous