Lecture 2, Sep 12, 2022

Simple Harmonic Motion

- Periodic motion about an equilibrium, caused by a restoring force
  - Can think of it as the 1D projection of circular motion
- $\vec{F} = -k\Delta\vec{x}$
  - The restoring force is conservative (gradient of a scalar field potential)
- SHM exhibits a sinusoidal pattern
- Applicable to many scenarios:
  - Mass on spring, rigid beams, crystals, chemical bonds, etc
- To get the DE: $F_x = -kx \implies -kx = ma_x \implies -kx = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \implies \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{k}{m}x = 0$
- Let $\omega^2 = \frac{k}{m}$ to restrict it to positive, so $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0$

Important

The general solution for SHM is $x(t) = X_0 + A\cos(\omega t + \phi_0)$, where:
- X_0 is an offset for the position at equilibrium
- $\omega = \sqrt{\frac{k}{m}}$
- Two constants to match initial conditions: A is the amplitude, ϕ_0 is the phase

- ω is the angular frequency in radians per second
  - $\omega = 2\pi f = \frac{2\pi}{T}$
- Can also write $x(t) = A\cos\left(2\pi\frac{t}{T} + \phi_0\right)$ in terms of the period instead of angular frequency
- Velocity is $v(t) = -A\omega\sin(\omega t + \phi_0)$
  - Max speed $v_{max} = A\omega$, occurs near equilibrium (centre of oscillation)
- Acceleration is $a(t) = -A\omega^2\cos(\omega t + \phi_0)$
  - Max acceleration $a_{max} = A\omega^2$, occurs near boundary of oscillation
  - Note this is also equal to $-\omega^2 x(t)$ which matches our equation of motion
- Alternative form: $x(t) = A\cos(\omega t + \phi_0) = A\cos(\omega t)\cos(\phi_0) - A\sin(\omega t)\sin(\phi_0)$
  - Since $A\cos(\phi_0), A\sin(\phi_0)$ are constants, $x(t) = a\cos(\omega t) + b\sin(\omega t)$

Important

The general solution can also be expressed as a sum of a sine and cosine: $x(t) = a\cos\omega t + b\sin\omega t$, where $a = A\cos(\phi_0), b = -A\sin(\phi_0)$

- Notice $a = A\cos(\phi_0)$ is simply $x(0)$, the initial position, and $b\omega = -A\omega\sin(\phi_0) = v(0)$
- Notice: $a^2 + b^2 = A^2(\cos^2\phi_0 + \sin^2\phi_0) = A^2$ and $\frac{b}{a} = -\tan\phi_0$

Note

The coefficients a and b of the sine and cosine are components of the oscillation's phasor $\boldsymbol{Z} = a - bi$