Lecture 13, Oct 6, 2022

Energy of a Wave

- The wave's energy consists of both kinetic and potential energy
  - The potential energy comes from the bending of the string against tension
  - If the string has linear mass density μ and tension τ and a small segment has unstretched (i.e. horizontal) length $\mathrm{d}x$, extended length $\mathrm{d}s$
  - $\mathrm{d}K = \frac{1}{2}\mu \mathrm{d}x \left(\frac{\partial y}{\partial t}\right)^2$
  - Potential energy derivation is much more ugly but $\mathrm{d}U = \frac{1}{2}\tau \mathrm{d}x \left(\frac{\partial y}{\partial x}\right)^2$
- Let's analyze the energy of the nth mode: using $y_n(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t)$ and integrate over the length of the string
- Total energy is $E = \frac{1}{4}\mu\omega_n^2 A_n^2 L$
  - Notice, μL is mass, and $A_n\omega_n$ is velocity of the oscillation
  - The $\frac{1}{4}$ is due to averaging
  - Note $\omega_n = \frac{n\pi v}{L}$ so ω_n^2 is proportional to n^2
- If we integrate to one wavelength $E = \frac{1}{2}\mu\omega_n^2 A_n^2 \frac{L}{n} = \frac{1}{4}\mu\omega_n^2 A_n^2 \lambda_n$
- The power is $P = \frac{E_n}{T} = \frac{1}{4}\mu\omega_n^2 A_n^2 v$
  - Recall $\mu v = Z$ is the impedance

Reflected Power

- The incident wave has power $Z_1 A_i^2 \omega^2$
- Reflected wave has $Z_1 A_r^2 \omega^2 = Z_1 A_i^2 R \omega^2$
- The average transmitted power is $Z_2 A_t^2 \omega^2 = Z_2 (A_i T)^2 \omega^2$
- The reflected power ratio is then $\frac{Z_1 A_i^2 R\omega^2}{Z_1 A_i^2 \omega^2} R^2$, sometimes defined as R_e
- The transmitted power ratio is then $T_e = \frac{Z_2}{Z_1}T^2$, *not* T^2!
- If we add $R^2 + \frac{Z_2}{Z_1}T^2$ we get 1, which shows conservation of energy
  - Therefore we can also define energy transmission coefficient as $T_e = 1 - R_e$

Summary

Recall the amplitude reflection and transmission coefficients $R \equiv \frac{A_r}{A_i}, T \equiv \frac{A_t}{A_i}$ where

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}, T = \frac{2Z_1}{Z_1 + Z_2}$$

The power reflection and transmission coefficients are

$$R_e \equiv \frac{P_r}{P_i} = R^2, T_e = \frac{P_t}{P_i} = \frac{Z_2}{Z_1}T^2$$

related by $T_e + R_e = 1$