Lecture 12, Oct 4, 2022

Fourier Analysis of Standing Waves

- A solution for standing waves follows $\omega_n = \dfrac{n\pi v}{L}$ (identical boundaries) or $\omega_n = \dfrac{n\pi v}{2L}$ (mixed boundaries, odd n only)
  - Boundary at $x = 0$ determines the type of spacial function (sin or cos)
  - Boundary at $x = L$ determines the allowed modes
  - The displacement is described by $y(x,t) = (A_n \sin(k_n x) + B_n \cos(k_n x))\cos(\omega_n t)$
- Superposition principle: adding two waves together still gives us another wave
  - This means $Y(x,t) = \sum_n \cos(\omega_n t)(A_n \sin(k_n x) + B_n \cos(k_n x))$, the superposition of all standing waves, is a valid wave
  - Let's use only sines, then in general $y(x,t) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t)$
- Any initial shape of the string $y(x,0) = \sum A_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$ can be written as such a superposition; to determine the actual values of A_n we need, we can apply Fourier analysis
- Notice the properties:
  - $\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x = \frac{L}{2}$
  - $\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$
- Then we can multiply both sides of the superposition by $\sin\left(\frac{m\pi}{L}x\right)$ and integrate, to get $\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x = \int_0^L \sum_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x = A_n \frac{L}{2}$

Important

The coefficients A_n for the Fourier series

$$\sum A_n \sin\left(\frac{n\pi}{L}x\right)$$

can be obtained by

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x$$

Where:

1. n is the mode number
2. A_n is the coefficient, or the amplitude of the nth mode
3. L is the length of the string
4. $f(x)$ is the function describing the standing wave pattern observed

Note the integral is over the whole system, so the bounds may change depending on the setup of the system