Lecture 11, Oct 3, 2022

Standing Waves

- Standing waves occur when the system is constrained in some form (fixed or free ends)
- The simplest standing wave just has the entire string oscillate up and down together (half wavelength)
  - Particles in the middle move a lot, with little pressure change; particles near the end don't really move but there is a lot of pressure change
- The next standing wave has one entire wavelength
  - The amplitude decreases
  - Frequency doubles from the first mode
- The next mode has 1.5 wavelengths, then 2 wavelengths, and so on
  - All these modes can exist at the same time
- If the ends are open then the entire pattern shifts and the most displacement happens at the ends instead (pattern is simply shifted)
- If one end is closed and the other is open, we can't fit a half wavelength anymore, so the modes have $0.25, 0.75, 0.125$ wavelengths and so on
  - However between the first and second modes the frequency triples

Standing Waves Mathematically

- The solution is in the form of $y(x,t) = f(x)\cos(\omega t + \phi_i)$
  - $f(x)$ describes the amplitude along the wave
- Plugging the solution into the wave equation we get $v^2 \dfrac{\partial^2 f}{\partial x^2} = -\omega^2 f(x)$
  - $\omega = k = \dfrac{2\pi}{\lambda}$
- The general solution must be $f(x) = A\sin(kx) + B\cos(kx)$
  - With the boundary condition that $x(0) = 0$ and $x(L) = 0$ we have $B = 0$
  - This gives us discrete solutions $\omega_n = \dfrac{n\pi v}{L}$ for a string fixed at both ends
  - $y(x,t) = A_n \sin\left(\dfrac{n\pi}{L}x\right)\cos(\omega_n t)$
- With both ends fixed $k_n = \dfrac{\pi n}{L}, \lambda_n = \dfrac{2L}{n}$
  - First mode fits a half wavelength (fundamental frequency)
  - Second mode fits a full wavelength (2nd harmonic or 1st overtone)
  - Difference in frequency between consecutive harmonics is $\Delta f = f_1 = \dfrac{v}{2L}$
  - $f_n = nf_1$
- For longitudinal standing waves $\Delta x(x,t) = A_{SW}\sin(kx)\sin(\omega t)$
  - What we actually hear is not Δx but ΔP
  - Pressure and particle displacement are 90 degrees out of phase, i.e. pressure change is maximum where displacement is minimum
  - Displacement nodes are always locations of pressure antinodes
- For a system open at both ends $x(0) = y_0, x(L) = y_0 \implies A = 0$
  - $y(x,t) = A_n \cos\left(\dfrac{n\pi}{L}x\right)\cos(\omega_n t), \omega_n = \dfrac{n\pi v}{L}$
- If the boundaries are not perfect you'd get a superposition of the two
- Frequencies for a system open on both ends have the same frequencies as the closed boundary one but shifted so that displacement is maximum on the ends
- For mixed boundaries, only odd harmonics are present
  - Fundamental frequency has a quarter of a wavelength $f_1 = \dfrac{v}{4L}$
  - Next is the third harmonic, then the 5th harmonic and so on