Lecture 10, Sep 29, 2022

Waves

- A wave is a function of position and time: $y(x, t)$
- Ultimately a wave is nothing but a large number of coupled oscillators next to each other
- A travelling wave has $y(x, t) = f(x \pm vt)$
  - Since t only increases, but the wave can travel left or right, the sign of vt decides the direction of propagation
    - As x increases, we take away vt so the shape stays the same
  - A wave travelling to the right has the sign of vt negative
- Types of waves:
  - Mechanical waves require a medium of propagation
    - Transverse waves are waves where the displacement is perpendicular to the direction of travel, e.g. wave in a string
    - Longitudinal waves are waves where the displacement is parallel to the direction of travel, e.g. sound waves
      - In general these travel a little faster
  - Electromagnetic waves are self sustaining oscillations of the electromagnetic field; these require no medium so can travel through vacuum
    - These can also behave like particles!
  - Matter waves are the wavelike behaviour of small particles such as electrons

Sinusoidal Waves

- A sinusoidal wave is modelled by $y(t) = A\sin(kx \pm \omega t + \phi_0)$
  - ω is the angular frequency of the wave (if the wave is created by an oscillator, then that's the frequency of the oscillator)
  - k is the *angular wave number* $k = \dfrac{2\pi}{\lambda}$ where λ is the wavelength
    - k is also known as the *wave vector* (for a multidimensional wave it becomes a number)
    - Speed can be calculated: $v = \dfrac{\lambda}{T} = \lambda f = \dfrac{2\pi}{k}\dfrac{\omega}{2\pi} = \dfrac{\omega}{k}$
- $y(x, t) = A\sin\left(k\left(x \pm \dfrac{\omega}{k}t\right) + \omega_0\right) = A\sin\left(\dfrac{2\pi}{\lambda}(x \pm vt) + \phi_0\right)$
- We can either fix x and look at the history graph (where it looks like a normal oscillator, with a phase dependent on x), or fix t and look at a snapshot, which is a sinusoidal function in space
- Each particle has a changing velocity, therefore for waves to propagate a force is needed

The Wave Equation

- $y(x, t) = A\sin(kx \pm \omega t + \phi_0) \implies \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} = -A\omega^2\sin(kx \pm \omega t + \phi_0) = -\omega^2 y(x, t) \implies \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -Ak^2\sin(kx \pm \omega t + \phi_0) = -k^2 y(x, t)$
- Therefore $\dfrac{1}{k^2}\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{\omega^2}\dfrac{\partial^2 y}{\partial t^2} \implies \dfrac{\partial^2 y}{\partial x^2} = \dfrac{k^2}{\omega^2}\dfrac{\partial^2 y}{\partial t^2}$

Important

The Wave Equation: $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2}$

Wave Speed of a Stretched String

- Under a small angle approximation $v = \sqrt{\dfrac{\tau}{\mu}}$ where $\mu = \dfrac{m}{L}$ is the linear density and τ is the tension
- In general speed of a mechanical wave will always have a force element and a density element
  - For a solid the force element would be the Young's modulus

Mechanical Impedance

- A property of the medium that relates partial velocity to driving force: $Z = \dfrac{\tau_y(x,t)}{v_y(x,t)}$
  - Defines how much the medium resists motion when subjected to a force
- Define the mechanical impedance for:
  - Wave on a string: $Z = \sqrt{\mu\tau}$
  - Fluids: $Z_a = \sqrt{\rho B}$
  - Solid rod: $Z_a = \sqrt{\rho Y}$
- For fluids and solid rods this is also known as the acoustic impedance

Reflection and Transmission

- Consider when a wave travels from one medium to another with different Z
  - Two conditions: displacement is continuous: $f_1(x - c_1 t) + g_1(x + c_1 t) = f_2(x - c_2 t)$; slope is continuous: $\dfrac{\mathrm{d}}{\mathrm{d}x}(f_1(x - c_1 t) + g_1(x + c_1 t)) = \dfrac{\mathrm{d}}{\mathrm{d}x} f_2(x - c_2 t)$
  - This gives us $A_i + A_r = A_t$ and $A_i Z_1 - A_r Z_1 = A_t Z_2$

Definition

The reflection coefficient $R \equiv \dfrac{A_r}{A_i}$, the amplitude ratio of the reflected wave to the incident wave; the transmission coefficient $T \equiv \dfrac{A_t}{A_i}$, the amplitude ratio of the trasmitted wave to the incident wave

- Note amplitude ratios may be negative (e.g. $R < 1$ means the reflected wave is perfectly out of phase)

Important

The reflection/transmission coefficients may be calculated as:

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}, T = \frac{2Z_1}{Z_1 + Z_2} \implies 1 + R = T$$

- This means that $-1 \leq R \leq 1$ and $0 \leq T \leq 2$
- Consider the following cases:
  - Fixed boundary (wall): $Z_2 \to \infty \implies R = -1, T = 0$
    - Reflected wave is perfectly out of phase, nothing transmitted
  - Free boundary (ring on rod): $Z_2 = 0 \implies R = 1, T = 2$