Lecture 1, Sep 8, 2022

Lecture 2, Sep 12, 2022

Simple Harmonic Motion

- Periodic motion about an equilibrium, caused by a restoring force
  - Can think of it as the 1D projection of circular motion
- $\vec{F} = -k\Delta\vec{x}$
  - The restoring force is conservative (gradient of a scalar field potential)
- SHM exhibits a sinusoidal pattern
- Applicable to many scenarios:
  - Mass on spring, rigid beams, crystals, chemical bonds, etc
- To get the DE: $F_x = -kx \implies -kx = ma_x \implies -kx = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \implies \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{k}{m}x = 0$
- Let $\omega^2 = \frac{k}{m}$ to restrict it to positive, so $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0$

Important

The general solution for SHM is $x(t) = X_0 + A\cos(\omega t + \phi_0)$, where:
- X_0 is an offset for the position at equilibrium
- $\omega = \sqrt{\frac{k}{m}}$
- Two constants to match initial conditions: A is the amplitude, ϕ_0 is the phase

- ω is the angular frequency in radians per second
  - $\omega = 2\pi f = \frac{2\pi}{T}$
- Can also write $x(t) = A\cos\left(2\pi\frac{t}{T} + \phi_0\right)$ in terms of the period instead of angular frequency
- Velocity is $v(t) = -A\omega\sin(\omega t + \phi_0)$
  - Max speed $v_{max} = A\omega$, occurs near equilibrium (centre of oscillation)
- Acceleration is $a(t) = -A\omega^2\cos(\omega t + \phi_0)$
  - Max acceleration $a_{max} = A\omega^2$, occurs near boundary of oscillation
  - Note this is also equal to $-\omega^2 x(t)$ which matches our equation of motion
- Alternative form: $x(t) = A\cos(\omega t + \phi_0) = A\cos(\omega t)\cos(\phi_0) - A\sin(\omega t)\sin(\phi_0)$
  - Since $A\cos(\phi_0), A\sin(\phi_0)$ are constants, $x(t) = a\cos(\omega t) + b\sin(\omega t)$

Important

The general solution can also be expressed as a sum of a sine and cosine: $x(t) = a\cos\omega t + b\sin\omega t$, where $a = A\cos(\phi_0), b = -A\sin(\phi_0)$

- Notice $a = A\cos(\phi_0)$ is simply $x(0)$, the initial position, and $b\omega = -A\omega\sin(\phi_0) = v(0)$
- Notice: $a^2 + b^2 = A^2(\cos^2\phi_0 + \sin^2\phi_0) = A^2$ and $\frac{b}{a} = -\tan\phi_0$

Note

The coefficients a and b of the sine and cosine are components of the oscillation's phasor $\boldsymbol{Z} = a - bi$

Lecture 3, Sep 13, 2022

Simple Harmonic Oscillator Energy

- Since the force is conservative, we have a potential, which can be calculated by the negative work done by the oscillator
- Energy is conserved in a SHO: $K + U = c = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2$
  - Potential energy goes by $\cos^2$, kinetic energy goes by $\sin^2$
- Note the periodicity of energy is different (maximum kinetic/potential energy is reached twice per oscillation)

Damped Harmonic Oscillator

- Damping force at low speeds can be modelled as a drag force, proportional and opposing velocity
  - At higher speeds $F_d \propto v^2$ but this is much harder to model
- $F_{net} = -kx - bv \implies m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0$
  - b is the drag coefficient, depending on the medium
  - b is also known as the damping constant (but also this name is sometimes given to γ)
- Let $\omega_0^2 = \frac{k}{m}, \gamma = \frac{b}{m} \implies \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0$
  - ω_0 is the natural frequency of the oscillation (frequency if it were undamped)
  - The damped oscillator does not oscillate at the same frequency

Light Damping (Underdamping)

- Damping is light enough that oscillations continue, losing amplitude slowly
- Predict that solution looks like $x(t) = A_0 e^{-\beta t}\cos(\omega t)$ (ignore phase constant for now)
- Plugging this solution back in: $A_0 e^{-\beta t}\left((2\beta\omega - \gamma\omega)\sin(\omega t) + (\beta^2 - \omega^2 - \gamma\beta + \omega^2\cos(\omega t)\right) = 0$
  - This is only zero when $2\beta\omega - \gamma\omega = 0$ and $\beta^2 - \omega^2 - \gamma\beta + \omega_0^2$
  - Solve: $\beta = \frac{\gamma}{2}, \omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$
  - Notice that the new ω is smaller, since the damping slows it down

Important

The general solution for an underdamped oscillator: $x(t) = A_0 e^{-\frac{\gamma}{2}t}\cos(\omega t + \phi_0)$, where $\omega = \sqrt{\omega_0^2 + \frac{\gamma^2}{4}}$

Heavy Damping (Overdamping)

- Damping is heavy enough that no oscillations occur; system returns to equilibrium sluggishly
- We know the amplitude decays exponentially, but something else will also happen, so guess $x(t) = e^{-\beta t}f(t)$
- Plugging this back in leads to $\frac{\mathrm{d}^2f}{\mathrm{d}t^2} = \alpha^2 f, \alpha^2 = \frac{\gamma^2}{4} - \omega_0^2$
  - If $\omega_0^2 > \frac{\gamma^2}{4}$ we just have an underdamped oscillator
  - Otherwise $\alpha^2 > 0 \implies -\omega_0^2 > \frac{\gamma^2}{4}$, and $f(t) = Ae^{\alpha t} + Be^{\beta t}$

Important

The general solution for an overdamped oscillator: $x(t) = e^{-\frac{\gamma}{2}t}\left(Ae^{\alpha t} + Be^{\beta t}\right)$

Critical Damping

- When $\omega_0^2 - \frac{\gamma^2}{4} = 0$, the equation from heavy damping becomes $\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = 0$
- The general solution is $f(t) = A + Bt$, so $x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$
  - A has dimensions of length, B has dimensions of speed
- The behaviour is a smooth exponential decay

Important

The general solution for a critically damped oscillator: $x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$

Lecture 4, Sep 15, 2022

Damped Harmonic Oscillator Energy

- Recall in an underdamped oscillator $\omega_0^2 > \frac{\gamma^2}{4}$
- Because of the damping, energy is lost as heat
- Assuming $\frac{\gamma^2}{4} \ll \omega_0^2 \implies \omega \approx \omega_0$
  - $x(t) = A_0 e^{-\frac{\gamma}{2}t} \cos(\omega_0 t)$
  - $\frac{\mathrm{d}x}{\mathrm{d}t} = -A_0\omega_0 e^{-\frac{\gamma}{2}t} \sin(\omega_0 t)$, under the approximation that $\frac{\gamma}{2\omega_0}\cos(\omega_0 t)$ is tiny
  - This leads us to $E(t) = \frac{1}{2}kA_0^2 e^{-\gamma t}$
  - Notice energy decays twice as fast as amplitude

Definition

The lifetime of a damped oscillator is $\tau = \frac{1}{\gamma}$, so $E(t) = E_0 e^{-\frac{t}{\tau}}$

- Every cycle amplitude decays by $\exp(-\gamma T)$
- $\frac{\mathrm{d}E}{\mathrm{d}t} = mv\frac{\mathrm{d}v}{\mathrm{d}t} + kx\frac{\mathrm{d}x}{\mathrm{d}t} = v(ma + kx)$, but $ma = -kx - bv \implies ma + kx = -bv$ so $\frac{\mathrm{d}E}{\mathrm{d}t} = -bv^2$

The Q Factor

Definition

The quality factor is defined as $Q = \frac{\omega_0}{\gamma}$

- Q is the quality factor, a measurement of how good the oscillator is (how slowly it loses energy)
- For $\frac{\gamma}{2} \ll \omega_0$, every period $\frac{E(t_2)}{E(t_1)} = e^{-\gamma T}$
- With a series expansion for exp, $\frac{E(t_2 = t + T)}{E(t_1 = T)} = 1 - \gamma T$
  - $\frac{\Delta E}{E(t_1)} = -\gamma T$
  - $\frac{E(t_1) - E(t_2)}{E(t_1)} = \gamma T = \frac{2\pi\gamma}{\omega} = \frac{2\pi}{Q}$
  - This says the quality factor is the energy stored in the oscillator over the energy dissipated per radian

- The approximation $\omega \approx \omega_0$ is quite valid even for small values of Q
- $Q = \frac{1}{2}$ is the smallest Q possible, which corresponds to a critically damped oscillator
  - $\gamma = \frac{\omega_0}{Q} \implies \omega = \omega_0 \sqrt{1 - \frac{1}{4Q^2}}$

Summary

For a lightly damped harmonic oscillator ($\omega_0 \approx \omega$):

- $E(t) = \frac{1}{2} k A_0^2 e^{-\gamma t}$
- $A(t) = A_0 e^{-\frac{\gamma}{2} t}$
- $\frac{\mathrm{d}E}{\mathrm{d}t} = -bv^2$

Lecture 5, Sep 19, 2022

Driven Oscillators

- An oscillator is driven by another oscillation
- With time the frequency of the oscillator will match the driving frequency
  - Initially the movement is messy (transient response), but with time it tries to match the driver
  - There will always be a phase lag
- Maximum amplitude happens when the driving frequency matches the natural frequency of the oscillator
  - Further increasing the frequency at this point decreases the amplitude, even if the driving frequency is a multiple of the natural frequency

Undamped Forced Oscillation

- Equation of motion: $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = F_0 \cos(\omega t)$
  - Assume the spring moves according to $\xi(t) = \xi_0 \cos(\omega t)$, then $F_0 = k\xi_0$
- The solution is $x(t) = A(\omega)\cos(\omega t - \delta)$
  - δ is the phase lag
- Substituting into the equation of motion: $A(\omega)(-\omega^2 + \omega_0^2)\cos(\delta) = \omega_0^2 \xi_0, A(\omega)(-\omega^2 + \omega_0^2)\sin\delta = 0$
  - $\tan\delta = 0$, which gives us either $\delta = 0$ or $\delta = \pi$
    * With no damping, the system is either perfectly in phase or perfectly out of phase with the driver
  - $A(\omega)\left(1 - \frac{\omega^2}{\omega_0^2}\right)\cos\delta = \xi_0, A(\omega)\left(1 - \frac{\omega^2}{\omega_0^2}\right)\sin\delta = 0$
  - If $\delta = 0$: $A(\omega) = \frac{\xi_0}{1 - \frac{\omega^2}{\omega_0^2}}$ for $\omega < \omega_0$
    * Response is perfectly in phase
  - If $\delta = \pi$: $A(\omega) = \frac{-\xi_0}{1 - \frac{\omega^2}{\omega_0^2}}$ for $\omega > \omega_0$
    * Response is perfectly out of phase
- For an undamped system the amplitude goes to infinity as $\omega \to \omega_0$

Damped Forced Oscillation

- $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$
- Solution is the same $x = A(\omega)\cos(\omega t - \delta)$
- Plug back in and we get $\tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$

- As $\omega \to \omega_0, \tan\delta \to \infty \implies \delta \to \frac{\pi}{2}$
  - * In resonance the response is 90 degrees out of phase
- $A(\omega) = \frac{\xi_0\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$
  - Amplitude is the largest for $\omega = \omega_0$, but $A(\omega)$ will never really go to infinity
  - For $\omega \to 0, A(\omega) \to \xi_0 = \frac{F_0}{k}$ (non periodic force) and $\delta \to 0$
  - For $\omega \to \omega_0, A(\omega) \to \frac{\xi_0\omega_0}{\gamma}$ and $\delta \to \frac{\pi}{2}$
  - For $\omega \to \infty, A(\omega) \to 0$ and $\delta \to \pi$
- For $\omega < \omega_0$ the oscillations are in phase, but for $\omega > \omega_0$ the oscillations become out of phase
  - This is normally a very sharp transition without damping, but with more damping it smoothes out
- Note when there is dampening, the maximum amplitude is not guaranteed when $\omega = \omega_0$
  - The largest amplitude happens when $\sqrt{(\omega_0^2 - \omega^2) + (\omega\gamma)^2}$ is minimized
  - This works out to $\omega = \omega_0\sqrt{1 - \frac{\gamma^2}{2\omega_0^2}}$
  - The maximum amplitude always occurs shortly before ω_0 when there is damping

Forced Oscillation Power Absorbed

- $v(t) = -v_0(\omega)\sin(\omega t - \delta)$
  - $v_0 = \frac{\xi_0\omega_0^2\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$
- With dampening power is dissipated at a rate of bv^2
- The instantaneous power changes, so we can integrate for the average power over a period
- $\bar{P}(\omega) = \frac{bv^2}{2} = \frac{\omega^2 F_0^2\gamma}{2m\left((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right)}$
  - Note $\omega \to 0$ or $\omega \to \infty, \bar{P} \to 0$

Lecture 6, Sep 20, 2022

Average Power of a Forced Oscillator

- When average power is potted against $\frac{\omega}{\omega_0}$, max power occurs near resonance
  - From the curve of average power vs frequency, we can figure out damping
  - Full width at half height when plotted against ω is γ
- The shape of this curve gets more asymmetric with more damping (higher γ)
- Full width at half height is $\frac{\gamma}{\omega_0} = \frac{1}{Q}$
  - The smaller the γ the narrower the peak
- Assume $\omega_0 \approx \omega$, then the equation can be simplified
  - The max power can be determined to be $\frac{F_0^2}{2m\gamma}$

Example: RLC Circuit

- $\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{LC}q = \frac{\epsilon_0}{L}\cos(\omega t)$
- The solution gives an amplitude similar to that of the mass spring system

Transient Response

- Eventually the frequency of the driven oscillator matches the frequency of the driver

- The system has inertia, so initially the system is inclined to oscillate at its natural frequency
- In the initial period we have a sum of two oscillations, the driven and natural
- For damped oscillators the frequency ω_0 oscillations will dissipate at a rate depending on γ
  - This is called the *transient response*

Lecture 7, Sep 22, 2022

Simple Pendulum

- $-mg\sin(\theta) = L\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$
- Under a small angle approximation $-\frac{g}{L}\theta = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$
  - In this case "small" means $< 10°$
- $\theta(t) = \theta_{max}\sin(\omega t + \theta_0), \omega^2 = \frac{g}{L}, T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$
- For any arbitrary object, we can use the centre of mass point and pivot point and the moment of inertia
  - $\theta(t) = \theta_{max}\sin(\omega t + \theta_0)$
  - $\omega^2 = \frac{mgd}{I}$
  - Even though m appears in the equation, the frequency does not actually depend on mass since it cancels with the m term in I

Coupled Oscillators

- Consider two pendulums with masses m_A, m_B connected by a spring with constant k

Normal Modes

Definition

A normal mode is a mode of oscillation where every mass is oscillating at the same frequency and fixed phased relation

- Consider 2 cases:
  - Displacing them the same amount $x_A = x_B$
    - In this case the springs are unstretched; both masses oscillate in phase with the same frequency $\omega_1 = \sqrt{\frac{g}{L}}$
  - Displacing them in opposite directions $x_A = -x_B$
    - Assuming masses are the same, then $m_A\frac{\mathrm{d}^2x_A}{\mathrm{d}t^2} = -\frac{mg}{L}x_A - 2kx_A$
      - The masses are stretched by the same amount, so the total spring stretch is $2x_A$
    - Both masses oscillate with frequency $\omega_2 = \sqrt{\frac{g}{L} + \frac{2k}{m}}$
    - The masses are out of phase
- These are the *normal modes* of oscillation
  - "Normal" since they are linearly independent

Superposition of Normal Modes

- In general $x_A \neq \pm x_B$
- Assume the angles are small so the spring is horizontal
- Restoring force on m_A is $-\frac{mg}{L}x_A - k(x_A - x_B)$, force on m_B is $-\frac{mg}{L}x_B + k(x_A - x_B)$
  - This gives us a system of coupled differential equations

- Adding the two equations shows SHM with variable $x_A + x_B$ and frequency $\omega = \sqrt{\frac{g}{L}}$
- Subtracting the two equations shows SHM with variable $x_A - x_B$ and frequency $\omega = \sqrt{\frac{g}{L} + \frac{2k}{m}}$
  - The $\frac{2k}{m}$ term represents the coupling
- Let $x_A + x_B = q_1, x_A - x_B = q_2$, then $q_1(t) = C_1 \cos(\omega_1 t + \phi_1), q_2(t) = C_2 \cos(\omega_2 t + \phi_2)$ where $\omega_1 = \sqrt{\frac{g}{L}}, \omega_2 = \sqrt{\frac{g}{L} + \frac{2k}{m}}$
- Energy is $E = \frac{1}{2}m\left(\frac{\mathrm{d}x_A}{\mathrm{d}t}\right)^2 + \frac{1}{2}m\left(\frac{\mathrm{d}x_B}{\mathrm{d}t}\right)^2 + \frac{1}{2}\frac{mg}{L}(x_A^2 + x_B^2) + \frac{1}{2}k(x_A - x_B)^2$

$$= \frac{1}{4}m\left(\frac{\mathrm{d}q_1}{\mathrm{d}t}\right)^2 + \frac{1}{4}\frac{mg}{L}(q_1)^2 + \frac{1}{4}m\left(\frac{\mathrm{d}q_2}{\mathrm{d}t}\right)^2 + \frac{1}{4}\left(\frac{mg}{L} + 2k\right)(q_2)^2$$

Lecture 8, Sep 26, 2022

Spring-Coupled Masses

- Two equal spring oscillators coupled by another spring
- $\begin{cases} m\frac{\mathrm{d}^2 x_A}{\mathrm{d}t^2} = -kx_A + k(x_B - x_A) = kx_B - 2kx_A \\ m\frac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} = -kx_B - k(x_B - x_A) = kx_A - 2kx_B \end{cases}$
- Consider a normal mode where both masses have the same frequency, i.e. $\begin{cases} x_A(t) = A\cos(\omega t) \\ x_B)t) = B\cos(\omega t) \end{cases}$
  - Substitute into 1: $\frac{A}{B} = \frac{k}{2k - m\omega^2}$
  - Substitute into 2: $\frac{A}{B} = \frac{2k - m\omega^2}{k}$
  - Therefore $2k - m\omega^2 = \pm k \implies \omega_1 = \sqrt{\frac{k}{m}}, \omega_2 = \sqrt{\frac{3k}{m}}$
    - $\omega^2 = \frac{k}{m} \implies A = B$ – regular oscillation, middle spring inactive
    - $\omega^2 = \frac{3k}{m} \implies A = -B$ – effectively two springs in the system
- General mode would be a superposition: $\begin{cases} x_A(t) = C_1\cos(\omega_1 t) + C_2\cos(\omega_2 t) \\ x_A(t) = C_1\cos(\omega_1 t) + C_2\cos(\omega_2 t) \end{cases}$ where $\omega_1 = \sqrt{\frac{k}{m}}, \omega_2 = \sqrt{\frac{3k}{m}}$
- Example: Consider if $x_A(0) = A, x_B(0) = 0$:
  - $q_1 = x_A + x_B, q_2 = x_A - x_B \implies x_A = \frac{1}{2}(q_1 + q_2), x_B = \frac{1}{2}(q_1 - q_2)$
  - Consider $q_1 = C_1\cos(\omega_1 t), q_2 = C_2\cos(\omega_2 t)$; plug in $t = 0$ and solve to get $C_1 = C_2 = A$
  - $x_A = \frac{1}{2}(C_1\cos(\omega_1 t) + C_2\cos(\omega_2 t)) = \frac{1}{2}A(\cos(\omega_1 t) + \cos(\omega_2 t)) = \frac{1}{2}A\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)$
    - We can think about this as a system oscillating with frequency $\frac{\omega_1 + \omega 2}{2}$ (i.e. the average of the two), and amplitude $\frac{\omega_1 - \omega_2}{2}t$
    - This is known as a beating phenomenon

Extending to More Masses

- Let $x_A = A\cos(\omega t), x_B = B\cos(\omega t)$, substitute into the equations of motion

- Recall $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x$
- $\begin{cases} A(2k - m\omega^2)\cos(\omega t) = kB\cos(\omega t) \\ B(2k - m\omega^2)\cos(\omega t) = kA\cos(\omega t \end{cases} \implies \begin{cases} \dfrac{2k}{m}A - \dfrac{k}{m}B = A\omega^2 \\ \dfrac{2k}{m}B - \dfrac{k}{m}A = B\omega^2 \end{cases}$
- Write in matrix form: $\begin{bmatrix} \dfrac{2k}{m} & -\dfrac{k}{m} \\ -\dfrac{k}{m} & \dfrac{2k}{m} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$
  - This is an eigenvalue problem
  - Solving us gets the same ω as before
- Example: Two equal masses m suspended from identical springs of constant k each
  - Note we don't need to worry about gravity since it's only a constant offset to the equilibrium
  - Consider a displacement down
  - $m\dfrac{\mathrm{d}^2x_A}{\mathrm{d}t^2} = -kx_A + k(x_B - x_A) \implies \dfrac{\mathrm{d}x_A}{\mathrm{d}t} = -\dfrac{2k}{m}x_A + \dfrac{k}{m}x_B$
  - $m\dfrac{\mathrm{d}^2x_B}{\mathrm{d}t^2} = -k(x_B - x_A) = \dfrac{k}{m}x_A - \dfrac{k}{m}x_B$
  - Let $x_A = A\cos(\omega t), x_B = B\cos(\omega t)$
  - As matrix equation: $\begin{bmatrix} -\dfrac{2k}{m} & \dfrac{k}{m} \\ \dfrac{k}{m} & -\dfrac{k}{m} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$

Lecture 9, Sep 27, 2022

Multiple Coupled Oscillators

- For N coupled masses we need to solve for the eigenvalues of an $N \times N$ matrix
- For a symmetrical system we can always factor out $\omega^2 - 2\dfrac{k}{m}$
- For a system of 3 oscillators, the frequencies are $\omega^2 = \dfrac{2k}{m}$ or $\dfrac{k}{m}\left(2 \pm \sqrt{2}\right)$

Coupled Forced Oscillations

- Consider a system of coupled spring oscillators where the end is driven by $\xi(t) = \xi_0 \cos(\omega t)$
  - The mass at the end gets an extra term, but the other masses have the same equation of motion
- It turns out we just get the regular driven oscillator, now with $x_A + x_B$ and $x_A - x_B$
  - Resonance occurs near any of the normal mode frequencies
  - When driving frequency is close to ω_1 the masses oscillate in phase; when it's close to ω_2 they oscillate out of phase

Lecture 10, Sep 29, 2022

Waves

- A wave is a function of position and time: $y(x, t)$
- Ultimately a wave is nothing but a large number of coupled oscillators next to each other
- A travelling wave has $y(x, t) = f(x \pm vt)$
  - Since t only increases, but the wave can travel left or right, the sign of vt decides the direction of propagation
    * As x increases, we take away vt so the shape stays the same
  - A wave travelling to the right has the sign of vt negative
- Types of waves:
  - Mechanical waves require a medium of propagation

    * Transverse waves are waves where the displacement is perpendicular to the direction of travel, e.g. wave in a string
    * Longitudinal waves are waves where the displacement is parallel to the direction of travel, e.g. sound waves
      • In general these travel a little faster
  – Electromagnetic waves are self sustaining oscillations of the electromagnetic field; these require no medium so can travel through vacuum
    * These can also behave like particles!
  – Matter waves are the wavelike behaviour of small particles such as electrons

Sinusoidal Waves

- A sinusoidal wave is modelled by $y(t) = A\sin(kx \pm \omega t + \phi_0)$
  – ω is the angular frequency of the wave (if the wave is created by an oscillator, then that's the frequency of the oscillator)
  – k is the *angular wave number* $k = \dfrac{2\pi}{\lambda}$ where λ is the wavelength
    * k is also known as the *wave vector* (for a multidimensional wave it becomes a number)
    * Speed can be calculated: $v = \dfrac{\lambda}{T} = \lambda f = \dfrac{2\pi}{k}\dfrac{\omega}{2\pi} = \dfrac{\omega}{k}$
- $y(x,t) = A\sin\left(k\left(x \pm \dfrac{\omega}{k}t\right) + \omega_0\right) = A\sin\left(\dfrac{2\pi}{\lambda}(x \pm vt) + \phi_0\right)$
- We can either fix x and look at the history graph (where it looks like a normal oscillator, with a phase dependent on x), or fix t and look at a snapshot, which is a sinusoidal function in space
- Each particle has a changing velocity, therefore for waves to propagate a force is needed

The Wave Equation

- $y(x,t) = A\sin(kx \pm \omega t + \phi_0) \implies \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} = -A\omega^2\sin(kx \pm \omega t + \phi_0) = -\omega^2 y(x,t) \implies \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -Ak^2\sin(kx \pm \omega t + \phi_0) = -k^2 y(x,t)$
- Therefore $\dfrac{1}{k^2}\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{\omega^2}\dfrac{\partial^2 y}{\partial t^2} \implies \dfrac{\partial^2 y}{\partial x^2} = \dfrac{k^2}{\omega^2}\dfrac{\partial^2 y}{\partial t^2}$

Important

The Wave Equation: $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2}$

Wave Speed of a Stretched String

- Under a small angle approximation $v = \sqrt{\dfrac{\tau}{\mu}}$ where $\mu = \dfrac{m}{L}$ is the linear density and τ is the tension
- In general speed of a mechanical wave will always have a force element and a density element
  – For a solid the force element would be the Young's modulus

Mechanical Impedance

- A property of the medium that relates partial velocity to driving force: $Z = \dfrac{\tau_y(x,t)}{v_y(x,t)}$
  – Defines how much the medium resists motion when subjected to a force
- Define the mechanical impedance for:
  – Wave on a string: $Z = \sqrt{\mu\tau}$
  – Fluids: $Z_a = \sqrt{\rho B}$
  – Solid rod: $Z_a = \sqrt{\rho Y}$

- For fluids and solid rods this is also known as the acoustic impedance

Reflection and Transmission

- Consider when a wave travels from one medium to another with different Z
  - Two conditions: displacement is continuous: $f_1(x - c_1t) + g_1(x + c_1t) = f_2(x - c_2t)$; slope is continuous: $\frac{\mathrm{d}}{\mathrm{d}x}(f_1(x - c_1t) + g_1(x + c_1t)) = \frac{\mathrm{d}}{\mathrm{d}x} f_2(x - c_2t)$
  - This gives us $A_i + A_r = A_t$ and $A_iZ_1 - A_rZ_1 = A_tZ_2$

Definition

The reflection coefficient $R \equiv \frac{A_r}{A_i}$, the amplitude ratio of the reflected wave to the incident wave; the transmission coefficient $T \equiv \frac{A_t}{A_i}$, the amplitude ratio of the trasmitted wave to the incident wave

- Note amplitude ratios may be negative (e.g. $R < 1$ means the reflected wave is perfectly out of phase)

Important

The reflection/transmission coefficients may be calculated as:

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}, T = \frac{2Z_1}{Z_1 + Z_2} \implies 1 + R = T$$

- This means that $-1 \leq R \leq 1$ and $0 \leq T \leq 2$
- Consider the following cases:
  - Fixed boundary (wall): $Z_2 \to \infty \implies R = -1, T = 0$
    - Reflected wave is perfectly out of phase, nothing transmitted
  - Free boundary (ring on rod): $Z_2 = 0 \implies R = 1, T = 2$

Lecture 11, Oct 3, 2022

Standing Waves

- Standing waves occur when the system is constrained in some form (fixed or free ends)
- The simplest standing wave just has the entire string oscillate up and down together (half wavelength)
  - Particles in the middle move a lot, with little pressure change; particles near the end don't really move but there is a lot of pressure change
- The next standing wave has one entire wavelength
  - The amplitude decreases
  - Frequency doubles from the first mode
- The next mode has 1.5 wavelengths, then 2 wavelengths, and so on
  - All these modes can exist at the same time
- If the ends are open then the entire pattern shifts and the most displacement happens at the ends instead (pattern is simply shifted)
- If one end is closed and the other is open, we can't fit a half wavelength anymore, so the modes have $0.25, 0.75, 0.125$ wavelengths and so on
  - However between the first and second modes the frequency triples

Standing Waves Mathematically

- The solution is in the form of $y(x, t) = f(x) \cos(\omega t + \phi_i)$
  - $f(x)$ describes the amplitude along the wave

- Plugging the solution into the wave equation we get $v^2 \frac{\partial^2 f}{\partial x^2} = -\omega^2 f(x)$
  - $\omega = k = \frac{2\pi}{\lambda}$
- The general solution must be $f(x) = A\sin(kx) + B\cos(kx)$
  - With the boundary condition that $x(0) = 0$ and $x(L) = 0$ we have $B = 0$
  - This gives us discrete solutions $\omega_n = \frac{n\pi v}{L}$ for a string fixed at both ends
  - $y(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t)$
- With both ends fixed $k_n = \frac{\pi n}{L}, \lambda_n = \frac{2L}{n}$
  - First mode fits a half wavelength (fundamental frequency)
  - Second mode fits a full wavelength (2nd harmonic or 1st overtone)
  - Difference in frequency between consecutive harmonics is $\Delta f = f_1 = \frac{v}{2L}$
  - $f_n = nf_1$
- For longitudinal standing waves $\Delta x(x,t) = A_{SW}\sin(kx)\sin(\omega t)$
  - What we actually hear is not Δx but ΔP
  - Pressure and particle displacement are 90 degrees out of phase, i.e. pressure change is maximum where displacement is minimum
  - Displacement nodes are always locations of pressure antinodes
- For a system open at both ends $x(0) = y_0, x(L) = y_0 \implies A = 0$
  - $y(x,t) = A_n \cos\left(\frac{n\pi}{L}x\right)\cos(\omega_n t), \omega_n = \frac{n\pi v}{L}$
- If the boundaries are not perfect you'd get a superposition of the two
- Frequencies for a system open on both ends have the same frequencies as the closed boundary one but shifted so that displacement is maximum on the ends
- For mixed boundaries, only odd harmonics are present
  - Fundamental frequency has a quarter of a wavelength $f_1 = \frac{v}{4L}$
  - Next is the third harmonic, then the 5th harmonic and so on

Lecture 12, Oct 4, 2022

Fourier Analysis of Standing Waves

- A solution for standing waves follows $\omega_n = \frac{n\pi v}{L}$ (identical boundaries) or $\omega_n = \frac{n\pi v}{2L}$ (mixed boundaries, odd n only)
  - Boundary at $x = 0$ determines the type of spacial function (sin or cos)
  - Boundary at $x = L$ determines the allowed modes
  - The displacement is described by $y(x,t) = (A_n \sin(k_n x) + B_n \cos(k_n x))\cos(\omega_n t)$
- Superposition principle: adding two waves together still gives us another wave
  - This means $Y(x,t) = \sum_n \cos(\omega_n t)(A_n \sin(k_n x) + B_n \cos(k_n x))$, the superposition of all standing waves, is a valid wave
  - Let's use only sines, then in general $y(x,t) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t)$
- Any initial shape of the string $y(x,0) = \sum A_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$ can be written as such a superposition; to determine the actual values of A_n we need, we can apply Fourier analysis
- Notice the properties:
  - $\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x = \frac{L}{2}$
  - $\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$

- Then we can multiply both sides of the superposition by $\sin\left(\frac{m\pi}{L}x\right)$ and integrate, to get $\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x = \int_0^L \sum_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x = A_n\frac{L}{2}$

Important

The coefficients A_n for the Fourier series

$$\sum A_n \sin\left(\frac{n\pi}{L}x\right)$$

can be obtained by

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x$$

Where:
1. n is the mode number
2. A_n is the coefficient, or the amplitude of the nth mode
3. L is the length of the string
4. $f(x)$ is the function describing the standing wave pattern observed

Note the integral is over the whole system, so the bounds may change depending on the setup of the system

Lecture 13, Oct 6, 2022

Energy of a Wave

- The wave's energy consists of both kinetic and potential energy
  - The potential energy comes from the bending of the string against tension
  - If the string has linear mass density μ and tension τ and a small segment has unstretched (i.e. horizontal) length $\mathrm{d}x$, extended length $\mathrm{d}s$
  - $\mathrm{d}K = \frac{1}{2}\mu\mathrm{d}x\left(\frac{\partial y}{\partial t}\right)^2$
  - Potential energy derivation is much more ugly but $\mathrm{d}U = \frac{1}{2}\tau\mathrm{d}x\left(\frac{\partial y}{\partial x}\right)^2$
- Let's analyze the energy of the nth mode: using $y_n(x,t) = A_n\sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t)$ and integrate over the length of the string
- Total energy is $E = \frac{1}{4}\mu\omega_n^2 A_n^2 L$
  - Notice, μL is mass, and $A_n\omega_n$ is velocity of the oscillation
  - The $\frac{1}{4}$ is due to averaging
  - Note $\omega_n = \frac{n\pi v}{L}$ so ω_n^2 is proportional to n^2
- If we integrate to one wavelength $E = \frac{1}{2}\mu\omega_n^2 A_n^2\frac{L}{n} = \frac{1}{4}\mu\omega_n^2 A_n^2\lambda_n$
- The power is $P = \frac{E_n}{T} = \frac{1}{4}\mu\omega_n^2 A_n^2 v$
  - Recall $\mu v = Z$ is the impedance

Reflected Power

- The incident wave has power $Z_1 A_i^2\omega^2$
- Reflected wave has $Z_1 A_r^2\omega^2 = Z_1 A_i^2 R\omega^2$
- The average transmitted power is $Z_2 A_t^2\omega^2 = Z_2(A_i T)^2\omega^2$

- The reflected power ratio is then $\frac{Z_1 A_i^2 R\omega^2}{Z_1 A_i^2 \omega^2} R^2$, sometimes defined as R_e
- The transmitted power ratio is then $T_e = \frac{Z_2}{Z_1} T^2$, *not* T^2!
- If we add $R^2 + \frac{Z_2}{Z_1} T^2$ we get 1, which shows conservation of energy
  - Therefore we can also define energy transmission coefficient as $T_e = 1 - R_e$

Summary

Recall the amplitude reflection and transmission coefficients $R \equiv \frac{A_r}{A_i}, T \equiv \frac{A_t}{A_i}$ where

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}, T = \frac{2Z_1}{Z_1 + Z_2}$$

The power reflection and transmission coefficients are

$$R_e \equiv \frac{P_r}{P_i} = R^2, T_e = \frac{P_t}{P_i} = \frac{Z_2}{Z_1} T^2$$

related by $T_e + R_e = 1$

Lecture 14, Oct 11, 2022

Travelling Waves

- Particles in the wave undergo simple harmonic motion
- $v_y(x,t) = \pm A\omega \cos(kx \pm \omega t + \phi_0)$
- $a_y(x,t) = \mp A\omega^2 \sin(kx \pm \omega t + \phi_0)$
- To find a travelling wave's velocity and acceleration (of a medium particle), take $\frac{\partial}{\partial t}$
- Speed of wave in gas: $v = \sqrt{\frac{\gamma k_B T}{m}}$ where m is the atomic mass, or $\sqrt{\frac{\gamma RT}{M}}$ where M is the molar mass
  - $\gamma = 1.67$ for a monoatomic gas, 1.4 for a diatomic gas

Lecture 15, Oct 7, 2022

Energy and Power of a Travelling Wave

- To propagate a wave, we need a power source; we want to find this power
- Energy is carried by either kinetic or potential energy
- Power $P = Fv$ so $P(x,t) = -\tau \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}$
  - $P(x,t) = \sqrt{\mu\tau} A^2 \omega^2 \sin^2(kx - \omega t + \phi_0)$
- Average power is $\frac{1}{2}\sqrt{\mu\tau} A^2 \omega^2$
  - Max power is $\sqrt{\mu\tau} A^2 \omega^2$
  - Note $Z = \sqrt{\mu\tau}$ and $A\omega = v_{max}$
- Sound waves propagate in multiple directions, so we define *intensity* $I = \frac{P_{avg}}{S}$, average power per unit area
  - Intensity is given by $\frac{\sqrt{\rho B} A^2 \omega^2}{2} = \frac{\Delta p_{max}^2}{2\rho v} = \frac{\Delta p_{max}^2}{2\sqrt{\rho B}}$
  - In 2D, $I \propto \frac{1}{r}$

- In 3D, $I \propto \frac{1}{r^2}$; if power at the source is P, then $I = \frac{P}{4\pi r^2}$

Attenuation

- Energy is lost to the wave medium as heat as the wave passes through
- Rate of absorption is proportional to wave intensity
- $\frac{\mathrm{d}I}{\mathrm{d}x} = -\alpha I$ where α is the *attenuation coefficient*
  - $I(x) = I(x_0)e^{-\alpha(x-x_0)}$
  - Higher frequency gives a higher attenuation
- Attenuation and spreading are additive
  - Multiply by $\left(\frac{r_0}{r}\right)^{N-1}$ to add the spreading, where N is the number of dimensions
  - $I(r) = I(r_0)e^{-a(r-r_0)}\left(\frac{r_0}{r}\right)^{N-1}$
- Note usually attenuation is given per unit length, but sometimes it's given per wavelength

Intensity Level

- Intensity level is measured in decibels, $\beta = (10\text{dB})\log\left(\frac{I}{I_0}\right)$
- For us, every $10\times$ increase in intensity sounds twice as loud
- Human range of hearing is from $I_0 = 1 \times 10^{-12}\,\text{W/m}^2$ (threshold of hearing) to $I = 1 \times 10^{1}\,\text{W/m}^2$ (threshold of pain)
- To calculate decibels of (sound) intensity, use $I_0 = 1 \times 10^{-12}\,\text{W/m}^2$ (remember to use a base 10 log!)

Lecture 16, Oct 18, 2022

Superposition of Waves

- Monochromatic waves have only one wavelength/frequency
- Waves can be superimposed
- Superimposing two waves creates beat phenomena
  - The wave is a multiplication of the two waves
  - The first wave has the average wave number and frequency
  - The second wave has $\frac{1}{2}(k_2 - k_1)$ and $\frac{1}{2}(\omega_2 - \omega_1)$
- The higher frequency makes the carrier frequency in the beat; the lower frequency makes the modulating frequency
  - The beat frequency is $\frac{1}{2}|\omega_2 - \omega_1|$
- This is used in AM radio

Important

Although the beat frequency has $\omega = \frac{1}{2}(\omega_2 - \omega_1)$, the frequency we hear is effectively twice this amount, at $\omega_2 - \omega_1$

Dispersion

- What happens when not al l the waves travel with the same frequency?
- In dispersive media, v and ω are functions of k, and wave speed depends on frequency
- A pulse, made of a superposition of waves, spreads out as it travels as the different components travel at different speeds

- The relation $\omega(k)$ is the *dispersion relation*
- This can be seen in e.g. prisms where in a dispersive medium, the refractive index for different wavelengths is different
- Phase velocity is defined as $v = \frac{\omega_0}{k_0}$, the ratio of the average frequency and wave number
  - The phase velocity is how fast a disturbance travels
  - The velocity of the carrier wave
- Group velocity is defined as $v_g = \frac{\Delta\omega}{\Delta k} = \frac{\omega(k_2) - \omega(k_1)}{k_2 - k_1}$
  - The group velocity is how fast the envelope travels
  - The velocity of the modulating wave
  - $v_g = \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k=k_0}$
  - $v_g = v - \lambda\frac{\mathrm{d}v}{\mathrm{d}\lambda}$

Lecture 17 (2-1), Oct 20, 2022

Lecture 18 (2-2), Oct 24, 2022

Blackbody Radiation

- Total radiation from a body is the sum of reflection and absorption
- A *blackbody* is some object that absorbs all incoming light at all wavelengths, i.e. no reflection
  - In order to satisfy energy balance, this blackbody must radiate out power
  - Kirchhoff's Law: Emissive power is proportional to absorption coefficient
    - Therefore emissive power of a blackbody is a universal property determined only by temperature
- Intuitively we know that things glow when they're hot
  - The spectral function of a hot object has more low wavelength emissions
  - Total radiative energy is proportional to T^4 (Stefan-Boltzmann Law)
  - Peak wavelength is inversely proportional to temperature, i.e. $\lambda_{max}T$ is constant (Wien's Displacement Law)
- To approximate a blackbody experimentally, Wien + Lummer used a box with a tiny hole; when light goes in, it reflects around inside and has a negligible chance of coming back out

Blackbody Radiation Theory

- Spectral distribution function ρ is given by $\rho(\lambda, T) = \lambda^{-5} f(\lambda T)$, since $\lambda_{max}T$ is constant
- Wien guessed $\rho(\lambda, T) = \frac{c_1}{\lambda^5} e^{-\frac{c_2}{\lambda T}}$, which fits experimental results
  - Planck aimed to come up with a theory that explained this
- Discrepancy between Wien's model and experimental data is observed for longer wavelengths

Planck's Law

- Planck assumed the energy of blackbody radiation is quantized in the calculation of average energy
  - In the classical picture $\langle E \rangle = \int_0^\infty Ef(E)\,\mathrm{d}E$ where $f(E) = ce^{-\frac{E}{kT}}$ (Boltzmann)
    - This works out to kT, which does not match experimental data
  - Planck calculated $\langle E \rangle = \sum_0^\infty Ef(E)$
    - Assume $E = n(hf)$, i.e. energy is in quantized in units of some value proportional to frequency
    - This works out to $\frac{hf}{e^{-hf}kT - 1}$
- This matches Wien's model at lower wavelengths and explains the discrepancy at higher wavelengths due to the -1

- $\rho(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}$
- h is *Planck's constant*, $h = 6.626\,070\,15 \times 10^{-34}\,\mathrm{m^2\,kg/s}$
- Even though this quantization of energy was used to solve the blackbody radiation problem, no one really thought it was physically meaningful at the time – except for Einstein

Specific Heat of Solids

- Specific heat: amount of heat required to raise the temperature of matter by a unit amount
- Dulong-Petit Law: all solids have the same molar specific heat
- Diamond's specific heat is way lower than what the D-P law predicts
- Einstein solved this by assuming each oscillator in a solid has quantized energy levels, similar to the blackbody radiation problem

Lecture 19 (2-3), Oct 25, 2022

Photoelectric Effect

- Ultraviolet light causes metals to release electrons; electron energy does not depend on light intensity, but increases with frequency
  - The dependence of electron energy on frequency is unknown
- Einstein proposed the quantization of light, from Planck: $E_{max} = \frac{1}{2}mv_{max}^2 = hf - \phi$
  - This predicts that $E_{max} = eV_0$ is linear in f, with slope of h
  - These predictions were confirmed by Millikan
- Einstein quantized light in *photons*, but nobody accepted it at first even after the Millikan experiment

Compton's Experiment

- This is the experiment that made physicists accept photons
- Investigates what happens when light is shined at electrons
- In the classical picture, since light is the oscillation of an electric field, an electron will oscillate inside this field as light passes through; this oscillating charge will radiate out light at the same frequency
  - This is *Thompson Scattering*, where $\lambda = \lambda'$
- In the quantum picture, since light is a particle, when the photon collides with the electron both the photon and electron will change course based like in a classical collision experiment
  - Collision can be elastic or inelastic, i.e. λ of the photon can change to λ' as it loses energy, which is gained by the electron
  - This is *Compton Scattering*, when $\lambda \neq \lambda'$ is possible
- $E = hf \implies p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}$
  - Consider photon coming in with $E_0, \vec{p}_0$ hitting an electron with rest energy mc^2
  - The electron goes out with $\vec{p}_2$, photon goes out with $(E_0, \vec{p}_1)$, with angle change θ
  - Momentum and energy conservation: $\vec{p}_0 = \vec{p}_1 + \vec{p}_2, E_0 + mc^2 = E_1 + \sqrt{(mc^2)^2 + (\vec{p}_2 c)^2}$
    * $\sqrt{(mc^2)^2 + (\vec{p}_2 c)^2}$ is the relativistic electron energy (don't worry about this right now)
  - This gives $\lambda_0 - \lambda_1 = \frac{h}{mc}(1 - \cos\theta)$
    * $\frac{h}{mc}$ is a length scale, referred to as the *Compton wavelength*, 0.0242Å
    * Since this change is very small, to notice the change we need to start with a very small λ_0, so X-rays are used
- Results from Compton's experiment confirmed the quantum picture, which made people accept the existence of photons

Lecture 20 (2-4), Oct 37, 2022

Spectroscopy and Bohr's Model

- Burning stuff to see what colour it emits
- First there was Thomson's plum pudding model, then it was discovered that atoms are mostly empty space, so came Rutherford's model of electrons orbiting a nucleus
  - However the orbiting electrons should emit radiation, which causes them to lose energy and spiral into the nucleus with a lifetime of 10×10^{-8} s
- Bohr's model had some assumptions:
  1. Electrons are in a circular orbit (Rutherford model)
  2. The electrons are in stationary states/orbits, i.e. they don't radiate energy
  3. Radiation is emitted when electrons change orbits
- Bohr assumed angular momentum is quantized $L = mvr = n\hbar$ where n is a positive integer and $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant
- To solve for the energy of the electron, equate the centripetal force with the Coulomb force:
  - $F_{cent} = m\frac{v^2}{r} = \frac{ke^2}{r}^2 = F_{Coulomb}$
  - $k\frac{e^2}{r^2} = \frac{mv^2}{r} \implies ke^2 = (mvr)v = Lv = n\hbar v$
  - This gives: $v = \frac{ke^2}{n\hbar}, r = \frac{(n\hbar)^2}{kme^2}$

Lecture 21 (2-5), Oct 31, 2022

X-Ray Diffraction

- X-rays are reflected from atoms in the crystalline structure of solids
- Reflected x-rays from different layers can interfere constructively or destructively, leading to a diffraction pattern
- Bragg's Law: $n\lambda = 2d \sin\theta$ leads to constructive interference, where d is the spacing between layers of atoms
  - We can orient the crystal in different ways and get different values of d to figure out the arrangement of atoms
  - This can also be used to determine λ given known θ and d
- Powder diffraction: doing an effective average of all the d distances by having tiny particles of the crystal as a powder, which lets us do all orientations at once

Matter Waves

- de Broglie: if photons can behave like both a particle and a wave, can electrons?
- $\lambda = \frac{h}{p}$ for photons, so can other matter act like waves?
- What if electrons around a Bohr model acted like standing waves?
  - From this assumption we can derive Bohr's quantization idea that $mvr = L = n\hbar$
- $\lambda = \frac{h}{\sqrt{2mE}}$ because $E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$, for an electron this is about 1.23Å
- This idea was proven by doing diffraction on electrons

Lecture 22 (2-6), Nov 1, 2022

Tonomura Experiment

- Electron double-slit experiment shows interference fringes characteristic of waves

- Electrons are sent one at a time, somehow still show a wavelike distribution – how do the electrons already know where to go?
- Therefore matter does behave like a wave – but what equation governs this wave?

Wave Equation for a Matter Wave

- For both light and matter waves $E = h\nu = \hbar\omega$ and $p = \dfrac{h}{\lambda} = \hbar k$
- However their dispersion relations are different:
  - For photons $E = pc \implies c = \dfrac{E}{p} = \dfrac{\omega}{k}$, so ω and k are linear in relationship
  - For matter $E = \dfrac{p^2}{2m} + V \implies \omega \propto k^2$, so ω is quadratic in k
- The classical wave equation, $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial x^2}$, would not work for matter waves
  - Take $y = A\sin(kx - \omega t) \implies \dfrac{\partial^2 y}{\partial x^2} = k^2 y, \dfrac{\partial^2 y}{\partial t^2} = \omega^2 y$
  - This can only be satisfied if the relationship between ω and k is linear
  - Matter waves cannot satisfy this due to their different dispersion relation
    * For a matter wave, we have to differentiate ω only once, but that means we have to match a sine and a cosine
    * This suggests we use a complex exponential: $\Psi(x,t) = Ae^{i(kx-\omega t)}$
      - $\dfrac{\partial^2 \Psi}{\partial x^2} = (ik)^2 Ae^{i(kx-\omega t)} = -k^2\Psi = -\left(\dfrac{p}{\hbar}\right)^2 \Psi$
      - $\dfrac{\partial \omega}{\partial x} = (-i\omega)Ae^{i(kx-\omega t)} = -i\omega\Psi = -i\dfrac{E}{\hbar}\Psi$
      - We can relate the two: $\dfrac{\partial^2 \Psi}{\partial x^2} = -\dfrac{p^2}{\hbar^2}\Psi = -\dfrac{2m(E+V)}{\hbar^2}\Psi = -\dfrac{2mE}{\hbar^2}\Psi + \dfrac{2mV}{\hbar^2} = -\dfrac{2m}{\hbar}\left(i\hbar\dfrac{\partial \Psi}{\partial t}\right) + \dfrac{2m}{\hbar^2}V\Psi$
- This leads us to the (time-dependent) Schrödinger equation: $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar\dfrac{\partial \Psi}{\partial t}$
  - We can alternatively express this as $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2 \Psi}{\partial x^2} + V\Psi = E\Psi$, which is the time-independent Schrödinger equation

Properties of the Matter Wavefunction

1. $|\Psi(x,t)|^2 = P(x,t)$, the probability distribution/density
   - The probability of finding a particle in the interval $[x, x + \mathrm{d}x]$ is $|\Psi(x)|^2\,\mathrm{d}x = P(x)\,\mathrm{d}x$
2. The wavefunction is square-integrable
   - Since we're interpreting $|\Psi(x,t)|^2$ as the probability density, then it must be true that $\displaystyle\int_{-\infty}^{\infty} |\Psi|^2\,\mathrm{d}x = 1$ (normalization condition)
   - We satisfy this by scaling Ψ by a number, but this requires that the integral is finite and nonzero
3. Two waves can be superimposed and their sum will still satisfy the wavefunction
4. Ψ must be continuous

Lecture 23 (2-7), Nov 3, 2022

Interpretation of the Wave Function

- If we measure the particle to be at x at time t, where was the particle just before the measurement?
  - Was it there all the time? Is this unknown? Is this even a valid question?
- Copenhagen interpretation (Max Born)
  - The position is indeterminate until the measurement – it's not known

    - At the point of the measurement the wavefunction *collapses* to position x
    - The wavefunction before the measurement is a superposition of all the possible wavefunctions describing the particle at every possible point
    - This is opposed to the classical interpretation, which would say the particle would be at x even before the measurement
        * This would indicate the existence of hidden variables and the incompleteness of quantum mechanics
        * This was shown to be not true by Bell's inequality

Operators

- If we have $P(x,t) = |\Psi(x,t)|^2$ we can calculate $\langle x \rangle = \int x|\Psi(x,t)|^2 \, \mathrm{d}x = \int \Psi^* x \Psi \, \mathrm{d}x$
- In quantum mechanics observable quantities (e.g. position, momentum) are represented by *operators*
    - e.g. $\hat{x}$ is the position operator
- The expectation value of operator $\hat{O}$ is $\Psi^* \hat{O} \Psi \, \mathrm{d}x$
    - This is the value actually measured in an experiment
- The momentum operator $\hat{p}$, how do we define it?
    - Consider the time derivative of position $\frac{\mathrm{d}}{\mathrm{d}t}$
    - $\frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{x} \rangle = -\frac{\partial i\hbar}{\partial m} \int \Psi^* \frac{\partial \Psi}{\partial x} \, \mathrm{d}x = \frac{1}{m} \int \Psi^* \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi \, \mathrm{d}x$
    - This gives us the momentum operator $\hat{p} = i\hbar \frac{\partial}{\partial x}$
- Ehrenfest's Theorem: expectation values follow classical mechanics

Separation of Variables

- If we assume $\Psi(x,t) = \psi(x)\phi(t)$ we can plug this into the SE
    - This gives us $i\hbar \frac{1}{\phi} \frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V(x)$
    - The left hand side is a function of only time and the right hand side is a function of only position, so they must be constants
    - Let both equal E
- For the left hand side we get $i\hbar \frac{1}{\phi} \frac{\mathrm{d}\phi}{\mathrm{d}t} = E \implies \phi = e^{-\frac{iE}{\hbar} t}$
- For the right hand side we get $-\frac{\hbar^2}{2m} \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V\psi = E\psi$, which is the time-independent Schrödinger equation
    - Notice $-\frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is equal to $\frac{\hat{p}^2}{2m}$
    - Define this as $\hat{T} = \frac{\hat{p}^2}{2m}$, the kinetic energy operator
    - Define the potential energy operator $\hat{V} = V(x)$
    - Define the Hamiltonian as $\hat{T} + \hat{V}$
        * The Hamiltonian is the total energy and gives us energy conservation
        * $\langle \hat{H} \rangle = \int \psi^* \hat{H} \psi \, \mathrm{d}x = \int \psi^* E \psi \, \mathrm{d}x = E \int |\psi|^2 \, \mathrm{d}x = E$
    - This gives us $\hat{H}\psi = E\psi$
- Notice the TISE is an eigenvalue equation
    - For a given $V(x)$ and boundary conditions, there is a set of ψs that satisfy the equation with corresponding energy eigenvalue E
    - The boundary conditions lead to quantization in these solutions, which means that E, the total energy, is quantized
- Note $|\Psi(x,t)|^2 = |\psi(x)|^2$, which is why these states are called *stationary states*

Summary

The Hamiltonian is defined as

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) = \frac{\hat{p}^2}{2m} + \hat{V}$$

and the Time-Independent Schrödinger Equation can be formulated as

$$\hat{H}\psi(x) = E\psi(x)$$

The full time-dependent solution is then

$$\Psi(x,t) = \phi(t)\psi(x) = e^{-i\frac{E}{\hbar}t}\psi(x)$$

Lecture 24 (2-8), Nov 14, 2022

Infinite Square Well (1D)

- Bound state, localized in Δx
- $V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}$
  - The particle cannot be outside $(0, a)$ since the potential is infinite there
- Outside the well, $\psi = 0$ is the only thing that can satisfy the SE
- Inside the well: $\hat{H}\psi = E\psi \implies -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi = E\psi$
  - $\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{2mE}{\hbar^2}\psi = 0$
    - When $E < 0$: Let $\alpha^2 = -\frac{2mE}{\hbar^2} \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \alpha^2\psi = 0 \implies \psi(x) = Ae^{\alpha x} + Be^{-\alpha x}$
    - When $E > 0$: Let $k^2 = \frac{2mE}{\hbar^2} \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\psi = 0 \implies \psi(x) = A\sin(kx) + B\cos(kx)$
- Enforce continuity conditions: $\psi(x) = 0$ for $x < 0$ or $x > a$
  - $\psi(0) = 0 \implies B = 0$
  - $\psi(a) = 0 \implies A\sin(ka) = 0 \implies ka = n\pi \implies k = \frac{n\pi}{a}$
    - Assume A is nonzero because that leads to a non-normalizable solution
  - This gives $E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2 n^2}{2ma^2}$
  - Note we're not considering continuity of the derivative here because our potential is infinite
- Normalization condition: $\int_{-\infty}^{\infty} |\psi|^2\,\mathrm{d}x = 1 \implies A = \sqrt{\frac{2}{a}}$
- $\psi_n(x) = A\sin(k_n x) = A\sin\left(\frac{n\pi}{a}x\right) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$
- There are an infinite number of allowed solutions; $n = 1$ is the lowest energy state (ground state)
  - Notice it's not at zero energy! This lowest energy level is the *zero-point energy*
- The full solution is $\Psi_n(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{i\hbar\pi^2 n^2}{2ma^2}t}$
  - The most general solution is a sum of all of these: $\Psi(x,t) = \sum_{n=1}^{\infty} C_n\Psi_n(x,t)$
  - In general $\Psi(x,t)$ is not a stationary state

Uncertainty Principle

- Physically, why can't we have $n = 0$?
- Physically $E = 0 \implies p = 0 \implies \Delta p = 0$, so there is no uncertainty in momentum; this would mean $\Delta x \Delta p = 0$, violating the uncertainty principle
- What about $n = 1$?
  - Δx is at most a since the particle is never outside the box
  - Δp can be estimated as $2\sqrt{2mE} = 2\frac{\pi\hbar}{a}$
    - $p = \sqrt{2mE}$ is incorrect; energy is a vector and momentum is a scalar
      - In 1D this means we have $p = \pm\sqrt{2mE}$
  - $\Delta x \Delta p = 2\pi\hbar$

Lecture 25 (2-9), Nov 15, 2022

Copenhagen Interpretation Revisited

- The energy of a system is indeterminate until the measurement
  - Before the measurement, the wavefunction is a superposition of many eigenstates (e.g. $\psi_1, \psi_2, \psi_3, \cdots$)
- Measurements of the total energy will give the expectation value of $\hat{H}$
  - Each individual measurement will only give you $E_1, E_2, \cdots$ in quantized levels, but the probabilities of each energy makes it so that on average you measure the expectation value
- When the measurement happens, the wavefunction collapses to a certain eigenstate

ψ_n Form an Orthonormal Basis

- Orthonormality of ψ_n:
  - Define the inner product as $\int \psi_n^* \psi_m \,\mathrm{d}x$
  - Orthonormality means that $\int \psi_n^* \psi_m \,\mathrm{d}x = \delta_{mn}$ where δ_{mn} is the Kronecker delta
    - When $n = m$ since ψ is normalized this is trivially true
    - Consider the infinite square well, then $\int \psi_n^* \psi_m \,\mathrm{d}x = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\,\mathrm{d}x = \delta_{mn}$
- Completeness:
  - Any arbitrary function can be expressed as some linear combination of ψ_n
  - $f(x) = \sum_{n=1}^{\infty} c_n \psi_n$
  - For the infinite square well, this is $\sqrt{\frac{2}{a}}\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)$
    - This is a Fourier series, so this proves completeness
- Using these, we can show $\int \psi_m^* f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} c_n \int \psi_m^* \psi_n \,\mathrm{d}x = \sum_{n=1}^{\infty} \delta_{mn} = c_m$

Interpretation of c_n

- $\langle \hat{H} \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$
- $|c_n|^2$ is the "weight" of each energy

Lecture 26 (2-10), Nov 17, 2022

Free Particle

- Consider a free particle, i.e. $V(x) = 0$ everywhere
- $-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi = E\psi$
- Let $k^2 = \frac{2mE}{\hbar^2}$ and $\omega = \frac{E}{\hbar} = \frac{\hbar k^2}{2m}$
  - With these $\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\psi = 0$
  - Therefore $\psi = Ae^{ikx} + Be^{-ikx}$
- With this we can write the full solution as $\Psi(x,t) = \psi(x)e^{-\frac{E}{\hbar}t} = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$
  - This is a superposition of two travelling waves, one in to the left and one to the right
  - We can just take the first part and let $\Psi(x,t) = Ae^{i(kx-\omega t)}$
- Normalize: $\int |\psi|^2 \,\mathrm{d}x = \int A^2 \,\mathrm{d}x = \infty$, but this is not normalizable!
- $e^{i(kx-\omega t)}$ is a plane wave and it cannot be a wavefunction
- However, even though a single plane wave is not normalizable, if we superimpose many of them, the resulting *wave packet* becomes normalizable
- We can write $\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i(kx-\omega t)}\,\mathrm{d}k$
  - $\phi(k)$ is the analogue of c_n and k is the analogue of n, except that k is not quantized
- For time $t = 0$: (works for any other finite value of t)
  - $\phi(k)$ is the Fourier transform of $\Psi(x,0)$
  - $\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{ikx}\,\mathrm{d}x$
  - $\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}\,\mathrm{d}x$

Heisenberg Uncertainty Principle

- Notice that as $\phi(k)$ becomes more concentrated in k, $\psi(x)$ becomes more spread out in x
- For a wavepacket, Δx and Δk are inversely proportional: $\Delta x\Delta k \sim 1$
- Since $p = \hbar k$, we have $\Delta x\Delta p \sim \hbar$
- Often written as $\Delta x\Delta p \geq \frac{\hbar}{2}$
  - The exact relation is unimportant, but this is the fundamental limit
- We can also have time-energy uncertainty principle $\Delta t\Delta E \geq \frac{\hbar}{2}$
  - However this is different since t is not exactly an observable
  - Note if $\Delta E = 0$ we are in a stationary state, which has an infinite lifetime

Lecture 27 (2-11), Nov 21, 2022

Quantum Tunneling

- In classical mechanics, when a particle reaches a potential barrier greater than its energy, it will stop and turn back
- In the quantum case, because the wavefunction doesn't decay instantaneously, the particle have a finite probability to pass through a potential barrier
- This is known as *quantum tunneling*
- Conversely, in quantum mechanics even when the particle has enough energy to overcome a barrier, there is a finite probability that the particle is reflected instead

Potential Step

- Consider potential $V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases}$
- Strategy: break up the potential into two regions, separately solve TISE, then match boundary conditions
  - In this case we have continuity of $\psi(x)$ as well as $\psi'(x)$
  - A discontinuity in $V(x)$ leads to a discontinuity of $\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}$, but if $V(x)$ is finite, then $\frac{\mathrm{d}\psi}{\mathrm{d}x}$ will still be continuous
    * This is why we need to consider ψ' in this case, but not in the case of the infinite square well, because here $V(x)$ is finite but it's not in the infinite square well
- In the first region $x < 0 \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{2mE}{\hbar^2}\psi = 0 \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\psi = 0 \implies \psi_I(x) = Ae^{ikx} + Be^{-ikx}$
- In the second region $x > 0 \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{2m}{\hbar^2}(E - V_0)\psi = 0$
  - Define $\alpha^2 = \frac{2m}{\hbar^2}(V_0 - E)\psi \implies \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \alpha^2\psi = 0$
    * Note this is because we're considering when $E < V_0$
  - This gives $\psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x}$
- We know $C = 0$ otherwise ψ_{II} is not square integrable
- Matching initial conditions:
  - $\psi_I(0) = \psi_{II}(0) \implies A + B = D$
  - $\psi_I'(0) = \psi_{II}'(0) \implies Aik - Bik = -\alpha D$
- For now we're interested in the ratios $\frac{B}{A}$ and $\frac{D}{A}$
  - $1 + \frac{B}{A} = \frac{D}{A}$
  - $1 - \frac{B}{A} = -\frac{\alpha}{ik}\frac{D}{A}$
  - $2 = \left(1 - \frac{\alpha}{ik}\right)\frac{D}{A}$
  - $\left(1 + \frac{\alpha}{ik}\right) + \frac{B}{A}\left(1 - \frac{\alpha}{ik}\right) = 0$
  - $\frac{B}{A} = \frac{1 - \frac{i\alpha}{k}}{1 + \frac{i\alpha}{k}}$
  - $\frac{D}{A} = \frac{2}{1 + \frac{i\alpha}{k}}$
  - Note $\frac{\alpha}{k} = \sqrt{\frac{V_0 - E}{E}}$, so we can determine these by the ratio of the energies
- Note physically A is the wave coming in, B is the reflected wave, and D is the transmitted wave in the classically forbidden region
- Define the reflection coefficient $R = \frac{|B|^2}{|A|^2} = \left|\frac{B}{A}\right|^2 = \frac{1 + i\frac{\alpha}{k}}{1 - i\frac{\alpha}{k}}\frac{1 - i\frac{\alpha}{k}}{1 + i\frac{\alpha}{k}} = 1$
  - This means we have total reflection; even though when the reflection happens there's a finite probability of the particle in the forbidden region, eventually everything is reflected in the end

Lecture 28 (2-12), Nov 22, 2022

Potential Barrier

- If the potential step goes to 0 again we have a potential barrier
- $V(x) = \begin{cases} 0 & x < 0 \\ V_0 & 0 \leq x \leq a \\ 0 & x > a \end{cases}$

- In the third region we have $\psi_{III} = Fe^{ikx}$ (note there is no e^{-ikx} term since there will not be a wave moving to the left in this region)
- Inside the barrier $C = 0$ is no longer true because we can normalize it even with $C \neq 0$
- Calculate $R = \left|\frac{B}{A}\right|^2, T = \left|\frac{F}{A}\right|^2$
- After matching boundary conditions, $T = \frac{16E(V_0 - E)}{V_0^2}e^{-2\alpha a}$, assuming $V_0 \gg E$, where $\alpha^2 = \frac{2m(V_0 - E)}{\hbar^2}$
  - The expression for T and R in general are quite complicated, but if we take the limit $E \ll V_0$ then this simplifies
- The transmission coefficient is exponentially dependent on the barrier width and α
  - The wider the barrier, the harder tunneling is
  - The larger the barrier height $V_0 - E$ or mass m, the harder tunneling is
- In general with a potential barrier for any shape we can break it up into potential steps and integrate

Examples of Quantum Tunneling

- Field emission: consider electrons in a piece of metal in a vacuum;

Lecture 29 (2-13), Nov 24, 2022

Historical Roots of Relativity

- Galilean relativity is the first form, based on simple intuition
- Then Maxwell's equation was discovered an indicates that light travels at a constant speed c, based on universal constants
  - However this does not agree with Galilean relativity because what frame is c in?
  - Michelson-Morley experiment and many others confirm that the speed of light is constant, no matter the frame of reference
- Einstein's theory of relativity came to explain this

Definitions

- Reference frame: a particular perspective from which the universe is observed; a set of axes to measure position, momentum, etc and a clock to measure time
- Inertial reference frame: a reference in which there is no acceleration (constant velocity)
  - Special relativity deals with this
  - Any frame moving with a constant velocity wrt an inertial reference frame is also an inertial reference frame
  - Use v as the velocity of a reference frame and u as the velocity of something in that reference frame

Galilean (Classical) Relativity

- Intuitively if we throw a ball in a moving train we'd expect the ball to move with the train
- Under a Galilean transformation, if a reference frame is moving with velocity v in the x direction, then

$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$$

- Also $\begin{cases} u'_x = u_x - v \\ u'_y = u_y \\ u'_z = u_z \end{cases}$
- This means $a_x = a'_x$, i.e. acceleration is the same in all reference frames, so Newton's laws are the same in all frames
- However, this does not work for light, because under a Galilean transformation we can get light moving at greater than c in a reference frame, which is experimentally false
  - Early physicists tried to reconcile this with the idea of the aether which is the medium of EM waves, where the speed of light is c
    * In all other moving reference frames the speed of light will be different
  - If the aether exists, then the Earth must be moving through it, so the speed of light will be different from c ("aether wind")
  - The Michelson-Morley experiment proved this to be false
    * Gravitational wave detection (LIGO) works in the same way

Special Relativity

- Two postulates:
  1. The laws of physics are the same in each inertial reference frame
     - This does not mean that physical quantities are observed to be the same across all frames, but that laws of physics are followed by these quantities in the same way
  2. The speed of light is the same in all reference frames and nothing can go faster than the speed of light
- An event is something that happens that can be observed
  - Different observers can assign different spacetime coordinates to the same event
- A consequence is that simultaneity is not the same across two different reference frames

Lecture 30 (2-14), Nov 28, 2022

Special Relativity Definitions

- Define two dimensionless quantities, β and γ
  - $\beta = \frac{v}{c}$
  - $\gamma = \frac{1}{1-\beta^2} = \frac{1}{1-\frac{v^2}{c^2}}$
  - γ is close to 1 when velocity is not close to c; when v approaches c γ grows very quickly and diverges
    * When $\gamma \approx 1$ the problem is non-relativistic
  - When $\beta \ll 1$ we can use the Taylor expansion and write $\gamma = (1-\beta^2)^{-\frac{1}{2}} \approx 1 + \frac{\beta^2}{2}$
- Usually time is plotted on the vertical axis and space on the horizontal axis
  - If we plot out the trajectory of light in both directions we get two lines
  - Define a *light cone* as the region between the lines of the trajectory of light
    * Events in the negative light cone can influence events at the origin
    * Events at the origin can influence events in the positive light cone
  - When we take this to multiple dimensions, the "light triangle" becomes a light cone as we introduce other dimensions
- We scale the axes of the spacetime diagram so that they have the same units, so that the light cone lines have slope 1
- On a spacetime diagram within a light cone we can draw *world lines*, the trajectory of an object in spacetime
  - Nothing (starting at the origin) can be outside the light cone since that means it'd be going faster than the speed of light

The Light Clock and Time Dilation

- Consider reference frames s and s' moving horizontally with velocity v; an experiment is happening inside s' where an observer reflects light between two mirrors a height of h apart
- In s', $\Delta t' = \dfrac{2h}{c}$
- In s, light has to travel diagonally since s' is moving horizontally relative to s
  - When light is reflected from the first mirror, s' has moved by an amount $\dfrac{v(\Delta t)}{2}$; when the light is detected again s' moves another amount
    - Since light has to travel a longer distance, but the speed of light is the same, so time must increase!
  - $\left(\dfrac{c\Delta t}{2}\right)^2 = \left(\dfrac{v\Delta t}{2}\right)^2 + h^2 \implies \Delta t^2(c^2 - v^2) = 4h^2 \implies \Delta t = \dfrac{2h}{\sqrt{c^2 - v^2}} = \dfrac{2h}{c\sqrt{1 - \frac{v^2}{c^2}}} = \dfrac{2h}{c}\gamma = \gamma\Delta t'$
  - $\gamma > 1$ so $\Delta t' > \Delta t$, i.e. a time interval measured in s' is shorter than a time interval measured in s
  - Time has been "dilated" for the observer outside s'
- The observer in s' records the *proper time*; everyone else records a dilated time
  - $\Delta t'$ is the proper time since it's the time measured at the same place as the event, often written as Δt_0
- $\Delta t = \gamma\Delta t_0$
- We see evidence of this in muon decay

Lecture 31 (2-15), Nov 29, 2022

Length Contraction

- Consider reference frames s and s' moving with velocity v in the x direction; we try to measure the length l of a train car in s'
- If we want to do this in s, we can take the time when the start of the train passes the observer, and another time when the end passes the observer
  - This gives us a Δt and $l = v\Delta t$
- In s', $l' = v\Delta t'$
- In this case Δt is proper time so $\Delta t' = \gamma\Delta t \implies l' = \dfrac{l}{\Delta t}\Delta t' = \gamma l \implies l = \dfrac{l'}{\gamma}$
- The *proper length* is length as measured in a resting reference frame; in this case l' is the proper length
  - The observed length l is shorter, by a factor of $\dfrac{1}{\gamma}$
- $l = \dfrac{l_0}{\gamma}$ where l_0 is the proper length (length measured in the rest frame) and l is the length as observed from a moving frame

Lorentz Transformation

- The relativistic version of the Galilean transformation that addresses time dilation and length contraction
  - Connects $(x, y, z, t) \leftrightarrow (x', y', z', t')$
- Consider motion only in the x direction, between reference frames s and s' with constant velocity v
- Consider an even in s' with coordinates (x', t')
  - In the s frame $x = x' + vt$ under Galilean transformation, but x' should be length contracted so $x = \dfrac{x'}{\gamma} + vt$
  - In s' we have $\dfrac{x}{\gamma} = x' + vt'$
  - $\begin{cases} x' = \gamma(x - vt) \\ x = \gamma(x' + vt') \end{cases}$

- $x = \gamma(\gamma x - \gamma vt) + \gamma vt' = \gamma^2(x - vt) + \gamma vt'$
- $(1 - \gamma^2)x = -\gamma^2 vt + \gamma vt'$
- $t' = \dfrac{1-\gamma^2}{\gamma v}x + \gamma t = \gamma\left(t - \dfrac{v}{c^2}x\right)$

- The complete transformation: $\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \dfrac{v}{c^2}x\right) \end{cases}$
  - $\begin{cases} x = \gamma(x' + vt') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \dfrac{v}{c^2}x'\right) \end{cases}$
- What about velocity transformations?
  - Consider a particle with velocity u'_x, u'_y in s', what is $\vec{u}$, as measured from s?
  - $u_x = \dfrac{\Delta x}{\Delta t}$ and from the Lorentz transformation $\dfrac{\Delta x}{\Delta t} = \dfrac{\gamma(\Delta x + v\Delta t')}{\gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)} = \dfrac{\frac{\Delta x'}{\Delta t'} + v}{1 + \frac{v}{c^2}\frac{\Delta x'}{\Delta t'}} = \dfrac{u'_x + v}{1 + \frac{v}{c^2}u'_x}$
  - Similarly $u_y = \dfrac{\Delta y}{\Delta t} = \dfrac{\Delta y'}{\gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)} = \dfrac{u'_y}{\gamma\left(1 + \frac{v}{c^2}u'_x\right)}$

Lecture 32 (2-16), Dec 1, 2022

Relativistic Doppler Shift

- For classical waves the observed frequency is $f = \dfrac{f_0}{1 - \frac{v_{src}}{v_{wave}}}$ for a source moving towards an observer
- For a relativistic frequency shift, we need to take time dilation into account
  - $f = \dfrac{1}{\gamma}\dfrac{f_0}{1 - \frac{v_s}{v_w}} = f_0\sqrt{\dfrac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} = f_0\sqrt{\dfrac{1+\beta}{1-\beta}} = f_0\sqrt{\dfrac{c+v}{c-v}}$
    - For a source moving away, flip the signs
- Example: Redshift of stars suggests that they're moving away

Relativistic Momentum

- Consider a simple collision of two particles with mass m, colliding head-on with speed u towards each other; after the collision both particles change to the y direction
  - $\vec{u}_{1,i} = (u, 0, 0)$
  - $\vec{u}_{2,i} = (-u, 0, 0)$
  - Total momentum $\vec{p}_{tot} = 0$
  - After the collision the first particle becomes $\vec{u}_{1,f} = (0, u, 0)$ and the second particle becomes $\vec{u}_{2,f} = (0, -u, 0)$ so that $\vec{p}_{tot} = 0$ is conserved
- Now consider frame s' moving with velocity $v = u$ (i.e. a frame moving with particle 1 pre-collision)
  - Before the collision $\vec{u}'_{2i}$ is the only particle moving
  - The velocity of this particle would be $u'_x = \dfrac{u_x - v}{1 - \frac{u_x v}{c^2}} = \dfrac{-u-u}{1 - \frac{(-u)u}{c^2}} = \dfrac{-2u}{1 + \frac{u^2}{c^2}}$
  - $\vec{p}' = \dfrac{-2mu}{1 + \frac{u^2}{c^2}}\hat{x}$
  - After the collision, both particles have both an x and y component
  - $u'_{xf} = \dfrac{u_{xf} - v}{1 - \frac{u_{xf}v}{c^2}} = \dfrac{0-u}{1-0} = -u$
  - $u'_{yf} = \dfrac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} = \dfrac{u}{\gamma}$

- $\vec{p}' = m(-u)\hat{x} + m\frac{u}{\gamma}\hat{y} + m(-u)\hat{x} + m\frac{-u}{\gamma}\hat{y} = -2mu\hat{x}$
  - Momentum is seemingly not conserved, which violates our first postulates that all laws of physics stay the same
- Therefore we need to use a new definition of momentum
- Define the relativistic momentum $\vec{p} = \gamma_p m\vec{u}$ where $\gamma_p = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$
  - With this definition our momentum is now conserved

Lecture 33 (2-17), Dec 5, 2022

Relativistic Energy

- Use work-KE theorem
- $W = \int_i^f \vec{F} \cdot \mathrm{d}\vec{s}$
- Consider 1D movement from rest at $(x_0, 0)$ to (x, t) where it has velocity u; we apply a force $F(x)$ to move the particle by $\mathrm{d}x$
  - $W = \int_{x_0}^{x} F(x)\,\mathrm{d}x$
  - To find the force, we can start with the momentum since $F = \frac{\mathrm{d}p}{\mathrm{d}t}$
  - $\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{m}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\frac{\mathrm{d}u}{\mathrm{d}t}$
  - Also notice $u = \frac{\mathrm{d}x}{\mathrm{d}t}$ so $W = \int_{x_0}^{x} \frac{\mathrm{d}p}{\mathrm{d}t}\,\mathrm{d}x = \int_{x_0}^{x} \frac{m}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}x = \int_{x_0}^{x} \frac{m}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\frac{\mathrm{d}u}{\mathrm{d}t}u\,\mathrm{d}t = \int_{x_0}^{x} \frac{mu}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\,\mathrm{d}u$
  - The final result is $W = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} - mc^2 = mc^2(\gamma_p - 1)$
- The kinetic energy for a particle with velocity u is $K = mc^2(\gamma_p - 1)$
  - We can check that $K = 0$ for $u = 0$
  - In the limit $\frac{u}{c} \ll 1$, we can use a Taylor series to get $\gamma_p \approx 1 + \frac{u^2}{2c^2}$ so $K = \frac{u^2}{2c^2}mc^2 = \frac{1}{2}mu^2$, which is the classic kinetic energy
- We can rearrange this to get $\gamma_p mc^2 = K + mc^2$
  - mc^2 is a form of energy that's still there even when $u = 0$ – this is the *rest energy* of the particle, which is energy that comes from the mass of the particle alone
    * We can also think of mass as a form of "potential energy"
    * Example: the binding energy of a hydrogen atom is -13.6eV; this means that the hydrogen atom is actually lighter than $m_p + m_e$, by 2.5×10^{-35} kg
  - $\gamma_p mc^2$ is then the total energy
- $E = \gamma_p mc^2 = (\gamma_p - 1)mc^2 + mc^2$
- Now can we connect p and E?
  - $E^2 = (\gamma_p mc^2)^2 = \frac{m^2c^4}{1-\frac{u^2}{c^2}} = m^2c^4\left(\frac{1-\frac{u^2}{c^2}+\frac{u^2}{c^2}}{1-\frac{u^2}{c^2}}\right) = m^2c^4 + \frac{m^2u^2c^2}{1-\frac{u^2}{c^2}} = m^2c^4 + m^2c^4 = (pc)^2 + (mc^2)^2$
    * Therefore $E = \sqrt{(pc)^2 + (mc^2)^2}$
- We can also additionally define $\vec{\beta}_p = \frac{\vec{u}}{c} = \frac{\vec{p}c}{E}$
  - If we use energy units of eV, then we can use momentum units of eV/c and mass units of eV/c^2

- Notice $\frac{u}{c} = \frac{pc}{E}$, so if we know p or E, we can calculate the other, and then calculate $\frac{u}{c}$ to find u