Lecture 9, Sep 26, 2022

Nonhomogeneous to Homogeneous

- Homogeneous ODEs always have an equilibrium at the origin, whereas nonhomogeneous ODEs' equilibrium points aren't at the origin
- The equilibrium point for $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x} + b$ is $\vec{x}_{eq} = -A^{-1}b$
- With a change of coordinates $\tilde{\vec{x}} = \vec{x} - \vec{x}_{eq}$, we get $\frac{\mathrm{d}\tilde{\vec{x}} + \vec{x}_{eq}}{\mathrm{d}t} = A(\tilde{\vec{x}} + \vec{x}_{eq}) \implies \frac{\mathrm{d}\tilde{\vec{x}}}{\mathrm{d}t} = A\tilde{\vec{x}}$, a homogeneous system of ODEs

Superposition

- We like homogeneous ODEs because we can superimpose them

Important

Principle of Superposition: Given $\vec{x}_1(t), \vec{x}_2(t)$ are solutions to $\vec{x}'(t) = A\vec{x}(t)$, then $c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$ is also a solution for any c_1, c_2

- Proof: $\frac{\mathrm{d}}{\mathrm{d}t}(c_1\vec{x}_1(t) + c_2\vec{x}_2(t)) = c_1x_1'(t) + c_2x_2'(t) = c_1Ax_1(t) + c_2Ax_2(t) = A(c_1\vec{x}_1(t) + c_2\vec{x}_2(t))$

Linear Independence of Solutions

Definition

Two solutions $\vec{x}_1(t), \vec{x}_2(t)$ are linearly dependent if $\exists k$ s.t. $\vec{x}_1(t) = k\vec{x}_2(t)$

- Given two independent solutions, we can take linear combinations of them to span the full solution space and find a solution for any initial condition
- However, if the solutions are not independent, we can't do that

Definition

The Wronskian $W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix}\vec{x}_1(t) & \vec{x}_2(t)\end{vmatrix}$

If $W[\vec{x}_1, \vec{x}_2](t) = 0$, then x_1, x_2 are linearly dependent

General Solutions Through Eigendecomposition

- Given $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$, guess $\vec{x}(t) = e^{\lambda t}\vec{v}$
  - This guess corresponds to the straight line solutions; their directions don't change, and their magnitudes change exponentially
- Substituting in, we get $\lambda\vec{v} = A\vec{v}$: if λ and $\vec{v}$ are an eigenvalue and eigenvector of A, then $\vec{x} = e^{\lambda t}\vec{v}$ solves the ODE
- Assuming $\lambda_1 \neq \lambda_2$ we have two independent solutions $\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1, \vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2$ corresponding to the two eigenvalues and eigenvectors
  - We know the Wronskian is nonzero because eigenvectors for different eigenvalues are independent
- From them we can generate the general solution $c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$, which spans the 2D space of all initial conditions