Lecture 7, Sep 22, 2022

Logistic Growth

- Simple growth model of $\frac{\mathrm{d}y}{\mathrm{d}t} = rt$ is unrealistic, as at some point the population needs to stop growing due to lack of resources
- Growth rate depending on population: $\frac{\mathrm{d}y}{\mathrm{d}t} = h(y)y$
  - Growth rate $h(y) = r - ay$
  - If y is small, then $h(y) > 0$ and the population grows
  - If y is large, then $h(y) < 0$ and the population dies off due to lack of resources
- This leads to the logistic equation (Verhulst equation): $\frac{\mathrm{d}y}{\mathrm{d}t} = (r - ay)y$
  - Equivalently $\frac{\mathrm{d}y}{\mathrm{d}t} = r\left(1 - \frac{y}{K}\right)y, K = \frac{r}{a}$
  - This is a first-order autonomous nonlinear ODE
- r is the intrinsic carrying capacity
- K is the saturation level, or environmental carrying capacity

Important

Logistic growth model:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r\left(1 - \frac{y}{K}\right)y$$

where $K = \frac{r}{a}$; solved by

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

assuming $y_0 < K$

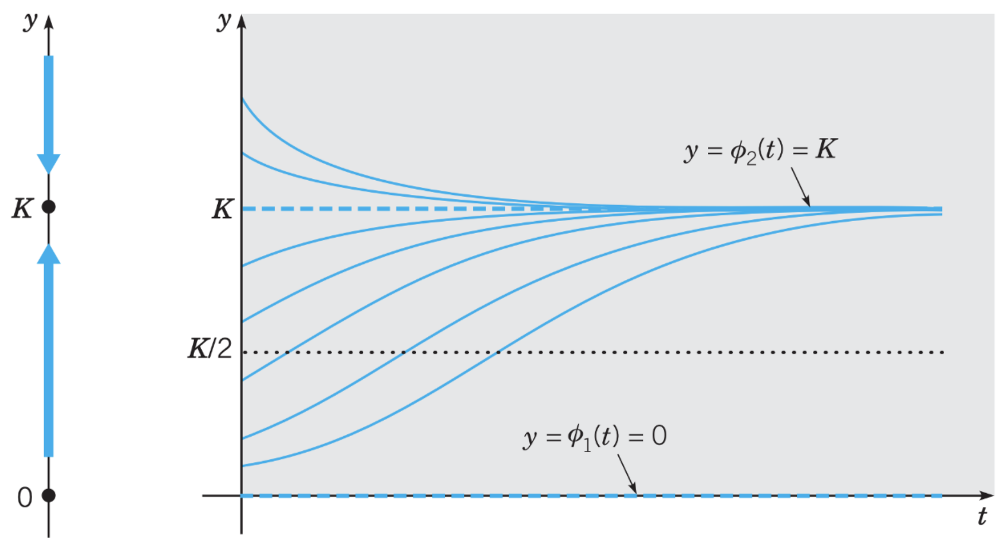

Figure 1: Solutions to the logistic model

- The line $y = K$ is a stable equilibrium
- $\frac{K}{2}$ is an inflection point, where the population curve goes from concave up to concave down
  - Rate of population growth begins to slow down

- The solution is $y = \dfrac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$, assuming $y_0 < K$

Growth With a Threshold

- If the initial population is too low, they might all die out before the population can grow
- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -r\left(1 - \dfrac{y}{T}\right)y$

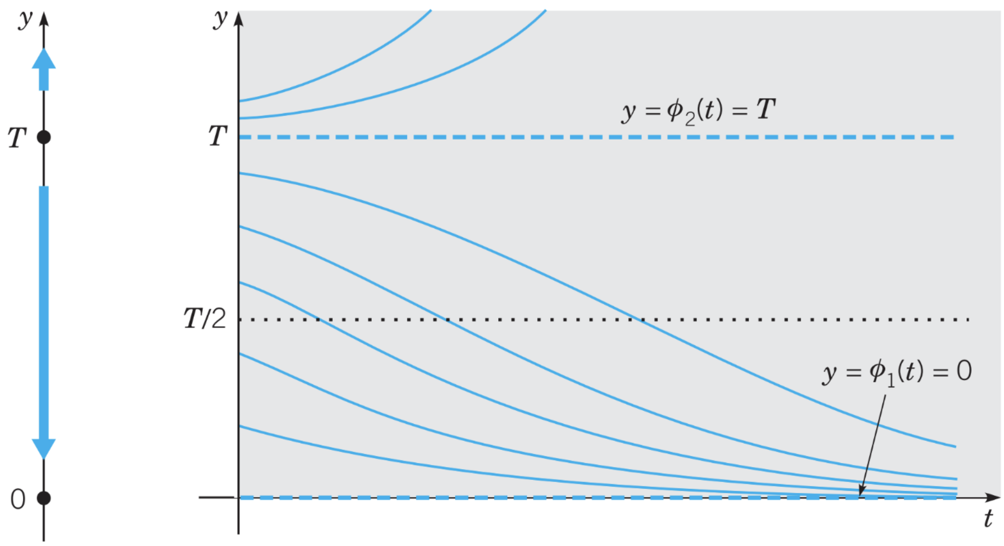

Figure 2: Solutions to the growth with a threshold model

- If $y_0 > T$ then the population keeps growing; if $y_0 < T$ then the population shrinks until everyone dies out
  - $y_0 = T$ is an unstable equilibrium
- Solution: $y = \dfrac{y_0 T}{y_0 + (T - y_0)e^{rt}}$

Logistic Growth With a Threshold

- Combine the two models: $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -r\left(1 - \dfrac{y}{T}\right)\left(1 - \dfrac{y}{K}\right)y$

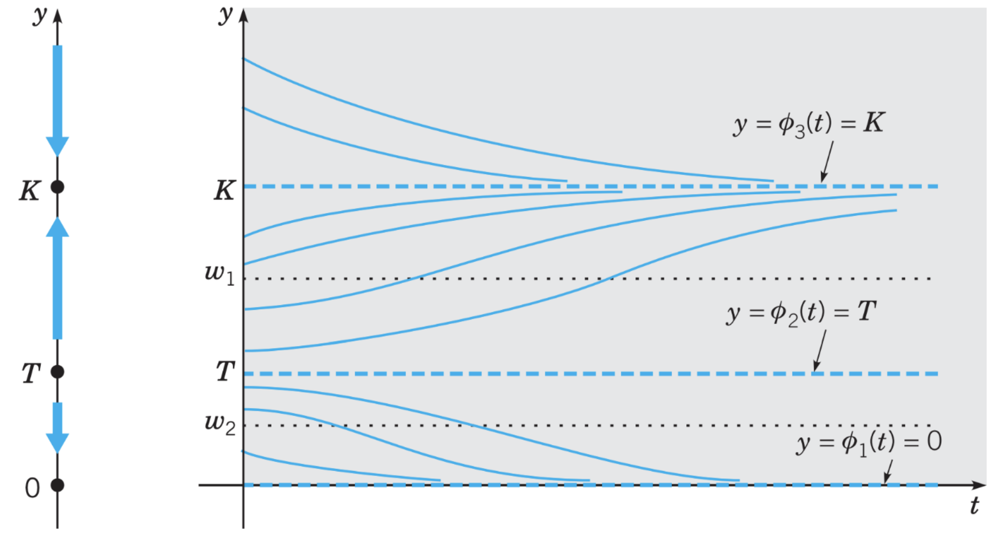

Figure 3: Solutions to the logistic growth with a threshold model