Lecture 6, Sep 19, 2022

Existence and Uniqueness of Solutions

- Given an ODE, does a solution exist, and is the solution unique?

Theorem

Given $y' + p(t)y = g(t), y(t_0) = y_0$, and p, g continuous over $t_0 \in (\alpha, \beta)$, then there exists a unique solution in the interval (α, β)

Theorem

Given $y' = f(t, y), y(t_0) = y_0$, and $f, \dfrac{\partial f}{\partial y}$ continuous over $(t_0, y_0) \in (\alpha, \beta) \times (\gamma, \delta)$, then there exists a unique solution in **some** interval $(t_0 - h, t_0 + h) \in (\alpha, \beta)$

The existence (but not uniqueness) of a solution can be established on the continuity of f alone

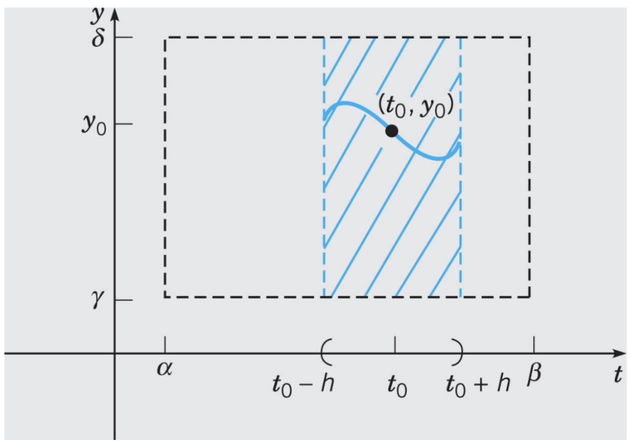

Figure 1: Visualization of the first-order nonlinear existence and uniqueness theorem

- The first-order nonlinear existence and uniqueness theorem only guarantees the existence and uniqueness of a solution in some interval within h, which we don't know
  - The linear version guarantees the entire continuous interval, whereas the nonlinear version only guarantees some smaller interval within the continuous interval
- Examples:
  - $y' + \dfrac{2}{t}y = 4t, y(1) = 2$
    * Use the linear theorem
    * p, g continuous except where $t = 0$
    * The initial condition lies within the continuous region, so a unique solution exists for $t \in (0, \infty)$
  - $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{3t^2 + 4t + 2}{2(y-1)}, y(0) = -1$
    * Use the nonlinear theorem
    * f is continuous except where $y = 1$
    * f_y is continuous except where $y = 1$
    * The initial condition lies in the region of continuity, so a unique solution exists for some region $t \in (-h, h)$
  - $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{3t^2 + 4t + 2}{2(y-1)}, y(0) = 1$

  * Since f is not continuous at the initial condition, the theorem does not apply
  * Even though it's not guaranteed that a solution will exist, a solution may still exist
  * Solving the DE does yield a solution, but the solution is not unique

Important

Implication: The graphs of two solutions cannot intersect each other where the theorems hold (because this would violate the uniqueness of solutions)

"Just because you can't see a solution to an ODE doesn't mean you can't prove it exists." – Vardan, MAT292 (2022)