Lecture 5, Sep 16, 2022

Rocket Science

- Consider Earth with radius R, rocket with mass m at height x with velocity v, and gravitational acceleration g
- Given $ma = F, v(0) = v_0$ as the initial rocket velocity
  - The force of gravity is $\frac{mgR^2}{(R+x)^2}$, since for $x = 0 \implies \frac{mgR^2}{R^2} = mg$ is the gravitational force on the Earth's surface
  - This gives $ma = F = -\frac{mgR^2}{(R+x)^2}$ (since gravity works in the negative direction)
- The equation of motion is $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{gR^2}{(R+x)^2}$
  - Applying the chain rule: $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}v$
  - Using this we can eliminate t
  - Note by doing this we get a first order ODE, whereas making $a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ gives us a second order ODE
- $v\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{gR^2}{(R+x)^2}, v(0) = v_0$
  - Note here $v(0) = v_0$ means the velocity at position 0 instead of time 0
- Solution:
$$\begin{aligned} v\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{gR^2}{(R+x)^2} \\ \implies \frac{1}{2}v^2 &= \frac{gR^2}{R+x} + C \\ \implies v(x) &= \pm\sqrt{\frac{2gR^2}{R+x} + C} \end{aligned}$$
  - $v(0) = v_0 \implies \sqrt{2gR + C} = v_0 \implies C = v_0^2 - 2gR$
  - Final solution: $v = \pm\sqrt{\frac{2gR^2}{R+x} + v_0^2 - 2gR}$
- What is the maximum altitude x_{max} reached?
  - $v(x_{max}) = 0 \implies \frac{2gR^2}{R+x} + v_0^2 = 2gR \implies x_{max} = \frac{v_0^2 R}{2gR - v_0^2}$
- Given x_{max}, what v_0 do we need?
  - $v_0 = \sqrt{2gR\frac{x_{max}}{R + x_{max}}}$
- The escape velocity is then $\lim_{x_{max}\to\infty} v_0 = \sqrt{2gR}$