Lecture 4, Sep 15, 2022

Method of Integrating Factors

- Consider a first order linear ODE: $\dfrac{\mathrm{d}u}{\mathrm{d}t} + p(t)u = g(t)$:
  - Assume we have a function $\mu(t)$ such that $\mu(t)p(t) = \dfrac{\mathrm{d}\mu}{\mathrm{d}t}$
  - Multiply by μ: $\mu(t)\dfrac{\mathrm{d}u}{\mathrm{d}t} + \mu(t)p(t)u = \mu(t)\dfrac{\mathrm{d}u}{\mathrm{d}t} + \dfrac{\mathrm{d}\mu}{\mathrm{d}t}u = \mu(t)g(t)$
  - $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mu u) = \mu(t)g(t) \implies u = \dfrac{\int \mu(t)g(t)\,\mathrm{d}t}{\mu(t)}$
  - Choose μ: $\dfrac{\mathrm{d}\mu}{\mathrm{d}t} = \mu(t)p(t) \implies \dfrac{1}{\mu}\dfrac{\mathrm{d}\mu}{\mathrm{d}t} = \mu(t)p(t) \implies \mu(t) = e^{\int p(t)\,\mathrm{d}t}$

Important

Method of integrating factors: The solution to $\dfrac{\mathrm{d}u}{\mathrm{d}t} + p(t)u = g(t)$ is $u(t) = \dfrac{1}{\mu}\left(\int \mu(t)g(t)\,\mathrm{d}t + C\right)$, where the integrating factor $\mu(t) = \exp\left(\int p(t)\,\mathrm{d}t\right)$

Example

- Example: $u' = -k(u - T_0 - A\sin(\omega t))$
  - The $A\sin(\omega T)$ term represents seasonal temperature variations
  - Put in standard form: $u' + ku = kT_0 + kA\sin(\omega t)$
  - Calculate integrating factor: $\mu = \exp\left(\int k\,\mathrm{d}t\right) = e^{kt}$
  - General solution: $u(t) = \dfrac{1}{e^{kt}}\left(\int e^{ks}(kT_0 + kA\sin(\omega s))\,\mathrm{d}s + C\right)$

$$
\begin{aligned}
&= T_0 + \frac{1}{e^{kt}}\left(\int e^{ks}kA\sin(\omega s)\,\mathrm{d}s\right) + C\frac{1}{e^{kt}} \\
&= \cdots \\
&= T_0 + \frac{kA}{k^2+\omega^2}(k\sin(\omega t) - \omega\cos(\omega t)) + C\frac{1}{e^{kt}}
\end{aligned}
$$

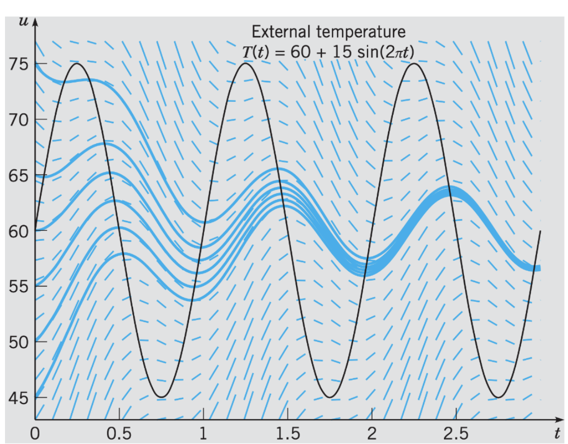

Figure 1: Solution curves

- Notice all solutions converge to a single solution $u(t) = T_0 + \dfrac{kA}{k^2 + \omega^2}(k\sin(\omega t) - \omega\cos(\omega t))$
  - The dominant term is completely independent of initial condition C
- However, this is not an equilibrium because $\dfrac{\mathrm{d}u}{\mathrm{d}t} \neq 0$