Lecture 33, Dec 5, 2022

Piecewise Continuous Functions as Vectors

- The set of piecewise continuous functions on (a, b) is denoted by $\mathrm{PC}[a, b]$
- This set is closed under scalar multiplication and addition, so it forms a vector space
- We define the *inner product* of two members of $\mathrm{PC}[a, b]$ as $\langle f, g\rangle = \int_a^b f(x)g(x)\,\mathrm{d}x$
  - The inner product is like a more generalized dot product
  - The usual properties of dot products are also satisfied: commutativity, linearity, distributivity, and $\langle f, f\rangle = 0$ if and only if $f = 0$
- Using the inner product we can define the *norm* as $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b (f(x))^2\,\mathrm{d}x}$
- $f, g \in \mathrm{PC}[a, b]$ are *orthogonal* if $\langle f, g\rangle = \int_a^b f(x)g(x)\,\mathrm{d}x = 0$

Definition

A set of functions

$$S = \{\, \phi_1(x), \phi_2(x), \cdots \,\} \in \mathrm{PC}[a, b]$$

are orthogonal if

$$\langle \phi_n, \phi_m\rangle = 0, n \neq m$$

and orthonormal if

$$\|\phi_n\| = 1, \forall n$$

i.e. $\langle \phi_n, \phi_m\rangle = \delta_{mn}$

Fourier's Theorem

- An important orthonormal set on $\mathrm{PC}[-L, L]$ is $\left\{ \sqrt{\frac{2}{L}}\frac{1}{2}, \sqrt{\frac{1}{L}}\sin\left(\frac{m\pi x}{L}\right), \sqrt{\frac{1}{L}}\cos\left(\frac{m\pi x}{L}\right), \cdots : m \in \mathbb{N} \right\}$
  - $\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,\mathrm{d}x = \delta_{mn}\pi$

Theorem

Fourier's Theorem: suppose f is periodic with period $2L$, $f, f' \in \mathrm{PC}[-L, L]$, then f can be expressed as a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right)$$

with Fourier coefficients given by

$$a_0 = \frac{1}{L}\langle f(x), 1\rangle \tag{1}$$

$$a_m = \frac{1}{L}\left\langle f(x), \cos\left(\frac{m\pi x}{L}\right)\right\rangle \tag{2}$$

$$b_m = \frac{1}{L}\left\langle f(x), \sin\left(\frac{m\pi x}{L}\right)\right\rangle \tag{3}$$

- The Fourier theorem is the direct analogue of the fact that you can represent a vector in another basis by taking the dot product of the vector with each of the basis vectors if the basis is orthonormal
  - A Fourier transform is nothing but a change of basis

- In the case of the discrete Fourier transform, $Ff = \hat{f}$, and due to orthonormality $F^T \hat{f} = f$