Lecture 32, Dec 2, 2022

Solving the Heat Equation

- We separated the system: $\begin{cases} X'' + \lambda X = 0 \\ T' + \alpha^2 \lambda T = 0 \end{cases}$
- In X: $X'' + \lambda X = 0$ with boundary conditions $X(0) = 0, X(L) = 0$
  - This gives us the general solution $X(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$
  - Using boundary conditions we have $c_1 = 0, c_2 \sin(\sqrt{\lambda} L) = 0$
  - This gives us solutions of $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ for integers n, giving $\lambda_n = \frac{n^2\pi^2}{L^2}$
  - Note $X(x)$ is the eigenfunction and λ is the eigenvalue of the $\frac{\partial^2}{\partial x^2}$ operator
- Substituting λ into the time ODE, we get $T(t) \propto \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2} t\right)$
- This gives us the set of fundamental solutions $u_n(x, t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2}{L^2} t}$
- The general solution is $u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2}{L^2} t}$
- With the initial condition $f(x) = u(x, 0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)$
- Now we can get c_n as the Fourier coefficients: $c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$