Lecture 31, Dec 1, 2022

Example

- System: $y'' + 2y' + 5y = g(t)$
- Transfer function: $H(s) = \dfrac{1}{s^2 + 2s + 5} = \dfrac{1}{(s+1)^2 + 4}$
- Impulse response: $h(s) = \mathcal{L}^{-1}\left\{H(s)\right\} = \dfrac{1}{2}e^{-t}\sin(2t)$
- Homogeneous solution: $\lambda = -1 \pm 2i \implies y_g(t) = e^{-t}(c_1\cos(2t) + c_2\sin(2t))$
- Particular solution (forced response): $h * g = \displaystyle\int_0^t \frac{1}{2}e^{-(t-\tau)}\sin(2(t-\tau))g(\tau)\,\mathrm{d}\tau$
- General solution: $y(t) = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t) + \displaystyle\int_0^t \frac{1}{2}e^{-(t-\tau)}\sin(2(t-\tau))g(\tau)\,\mathrm{d}\tau$

Partial Differential Equations (PDEs)

- Examples:
  - Heat equation $\dfrac{\partial u}{\partial t} = D\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$
    * Heat dispersing in an object
  - Wave equation: $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$
  - Laplace equation: $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$
- PDEs have more than one variable, whereas ODEs only have one variable x or t
- Consider 1D heat conduction along a cylinder, with temperature $u(x, t)$
- PDE: $u_t = a^2u_{xx}$, for $0 < x < L, t > 0$
  - α is the thermal diffusivity
  - Intuition: Points that are more "concentrated" in heat will have that heat spread out faster
  - Initial conditions: $u(x, 0) = f(x), 0 \leq x \leq L$
    * Initial temperature distribution
  - Boundary conditions: $u(0, t) = 0, u(L, t) = 0, t > 0$
    * Temperature at both ends of the rod is zero
- How would we approach something like this?
- Separation of variables: guess solution $u(x, t) = X(t)T(t)$
  - Substitution into ODE: $\dfrac{X''}{x} = \dfrac{1}{\alpha^2}\dfrac{T'}{T}$
  - Notice the left hand side is only a function of x while the right hand side is only a function of t, so for both to equal each other for all x and t, both must be constants
  - $\dfrac{X''}{X} = \dfrac{1}{\alpha^2}\dfrac{T'}{T} = -\lambda$
  - This gives us two ODEs: $\begin{cases} X'' + \lambda X = 0 \\ T' + \alpha^2\lambda T = 0 \end{cases}$