Lecture 30, Nov 28, 2022

Convolutions

Definition

The convolution of two functions $f(t)$ and $g(t)$ is

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$$

The discrete version for two sequences $f[n]$ and $g[n]$ is

$$(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m]g[n-m]$$

- Taking one function, flipping it around, shifting it by some amount, and seeing how much the two functions correlate
- Properties:
  - Commutativity: $f * g = g * f$
  - Distributivity: $f * (g_1 + g_2) = f * g_1 + f * g_2$
  - Associativity: $(f * g) * h = f * (g * h)$
  - Zero: $f * 0 = 0 * f = 0$
- Convolution in time domain is multiplication in s domain: $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$ (Convolution Theorem)
  - This also holds for the Fourier transform
  - Now if we have to take the inverse Laplace transform of some product, we can just use a convolution!

Input-Output Problem

- Consider the ODE $ay'' + by' + cy = g(t), y(0) = y_0, y'(0) = y_1$
  - Laplace transform: $(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_1 = G(s)$
  - $Y(s) = \dfrac{(as + b)y_0 + ay_1 + G(s)}{as^2 + bs + c}$
  - Let $H(s) = \dfrac{1}{as^2 + bs + c}$ be the *transfer function* of this system
  - $Y(s) = H(s)((as + b)y_0 + ay_1) + H(s)G(s)$
- We get the solution $y(t) = \mathcal{L}^{-1}\{H(s)((as + b)y_0 + ay_1))\} + \int_0^t h(t-\tau)g(\tau)\,\mathrm{d}\tau$
  - The first part of this, $\mathcal{L}^{-1}\{H(s)((as + b)y_0 + ay_1))\}$, is the *free response*
    - This is the response of the system to the initial conditions $y(0)$ and $y'(0)$ only, disregarding the forcing function
    - This can be thought of as the solution to the homogeneous system
  - The second part, $\int_0^t h(t-\tau)g(\tau)\,\mathrm{d}\tau$, is the *forced response*
    - This is the response of the system to $g(t)$ only, without any initial conditions (i.e. $y(0) = y'(0) = 0$)
    - This can be thought of as a particular solution to the non-homogeneous system
  - Combining the two we get the total response $y(t)$
- Note the forced response is $H(s)G(s)$; if we take $g(t) = \delta(t)$, then $H(s)G(s) = H(s)\mathcal{L}\{\delta(t)\} = H(s)$
  - The transfer function $H(s)$ is simply the impulse response of $ay''+by'+cy = \delta(t), y(0) = 0, y'(0) = 0$
  - Knowing the impulse response of the system allows us to easily determine the forced response