# Lecture 3, Sep 12, 2022

## Direction Fields

- Consider the DE $\frac{\mathrm{d}u}{\mathrm{d}t} = f(t, u)$
  - We can interpret this as the slope at each point is equal to some function of $t$ and $u$
- We can draw a direction field, at each point $(t, u)$ draw the slope $f(t, u)$

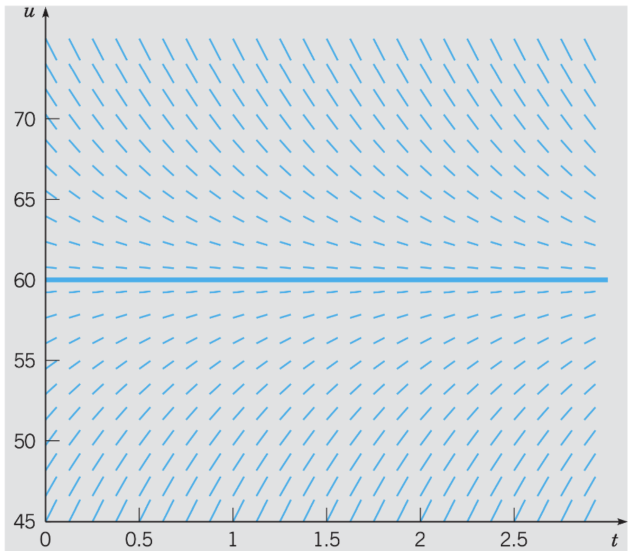


Figure 1: Direction field for $u' = -1.5(u - 60)$

- Using a direction field, for any starting point we can follow it to trace out a solution to our ODE

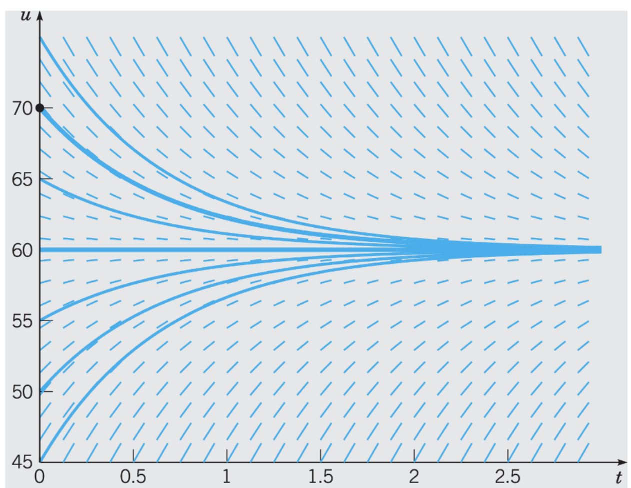


Figure 2: Direction field for $u' = -1.5(u - 60)$ with overlaid integral curves

- Direction fields allow us to visualize solutions to DEs without having to actually solve it

## Equilibria

- Notice for this DE, all solutions tend towards the **equilibrium** $u = 60$
  - If we start from the equilibrium, we never move away from it, which lends to the definition:

**Definition**

Given a first order autonomous DE $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$, equilibrium solutions are those satisfying $f(y) = \frac{\mathrm{d}y}{\mathrm{d}t} = 0$

Equilibrium points are also known as critical points, fixed points, stationary points, etc

**Definition**

3 types of equilibria:

- Stable equilibrium: other solutions tend towards the equilibrium solution
- Unstable equilibrium: other solutions diverge from the equilibrium solution
- Semi-stable equilibrium: other solution tend towards the equilibrium on one side and diverge from it on the other

- For this DE, we have a stable equilibrium since all solutions approach the equilibrium solution $u = 60$
- $\frac{\mathrm{d}p}{\mathrm{d}t} = rp - a$ has an unstable equilibrium of $p = \frac{a}{r}$, since all other solutions diverge from this point

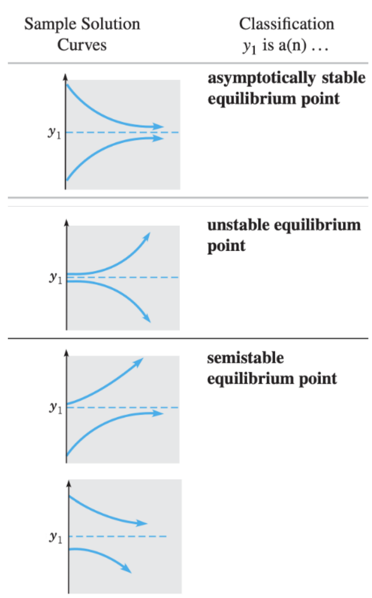


Figure 3: Types of eqilibrium

- Example: Find and classify equilibria of $y' = \cos y$:
  - $y' = 0 \implies \cos y = 0 \implies y = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$
  - The equilibrium at $\frac{\pi}{2}$ is stable, then $\frac{3\pi}{2}$ is unstable, $\frac{5\pi}{2}$ is stable, and so on
- On the plot, points where $y'$ crosses from positive to negative are stable; points where $y'$ crosses from negative to positive are unstable

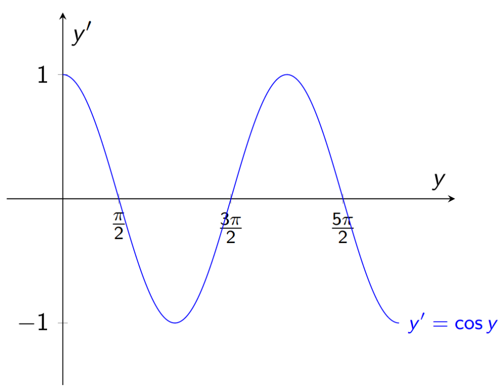


Figure 4: Plot of $y' = \cos y$