Lecture 29, Nov 21, 2022

Impulse Functions

- For some number ϵ we can define a delta epsilon function $\delta_\epsilon(t) = \dfrac{u_0(t) - u_\epsilon(t)}{\epsilon} = \begin{cases} \frac{1}{\epsilon} & 0 \leq t \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$
- The Dirac delta function is $\lim\limits_{\epsilon \to 0} \delta_\epsilon(t)$
  - At a single point, the value is infinite
  - The integral over the peak is 1
- $\mathcal{L}\left\{\delta_\epsilon(t)\right\} = \mathcal{L}\left\{\dfrac{u_0(t) - u_\epsilon(t)}{\epsilon}\right\} = \dfrac{1 - e^{-\epsilon s}}{\epsilon s}$
- Consider $y'' + y = I_0 \delta_\epsilon(t), y(0) = 0, y'(0) = 0$
  - $\mathcal{L}\left\{y'' + y'\right\} = (s^2 + 1)Y(s)$
  - $Y(s) = \dfrac{I_0}{\epsilon}(1 - e^{-\epsilon s})\dfrac{1}{s(s^2+1)}$
  - Let $H(s) = \dfrac{1}{s(s^2+1)} \implies h(t) = 1 - \cos(t)$
  - $Y(s) = \dfrac{I_0}{\epsilon}(H(s) - e^{-\epsilon s}H(s)) \implies y(t) = \dfrac{I_0}{\epsilon}\left(u_0(t)(1 - \cos(t)) - u_\epsilon(t)(1 - \cos(t - \epsilon))\right)$
  - $y_\epsilon(t) = \begin{cases} 0 & t \leq 0 \\ \dfrac{I_0}{\epsilon}(1 - \cos(t)) & 0 \leq t \leq \epsilon \\ \dfrac{I_0\left(\cos(t-\epsilon) - \cos(t)\right)}{\epsilon} & t > \epsilon \end{cases}$
  - $\lim\limits_{\epsilon \to 0} y_\epsilon(t) = y(t) = \begin{cases} 0 & t \leq 0 \\ -I_0 \dfrac{\mathrm{d}}{\mathrm{d}t}\cos(t) & t > 0 \end{cases} = \begin{cases} 0 & t \leq 0 \\ I_0 \sin(t) & t > 0 \end{cases}$
  - $y(t) = I_0 u(t) \sin(t)$

The Dirac Delta Function

Definition

The Dirac delta function is the function $\delta(t)$ with the following properties:

- $\delta(t - t_0) = 0$ whenever $t \neq t_0$
- $\displaystyle\int_a^b \delta(t - t_0)\,\mathrm{d}t = \begin{cases} 1 & a \leq t_0 \leq b \\ 0 & \text{otherwise} \end{cases}$
- $\displaystyle\int_a^b f(t)\delta(t - t_0)\,\mathrm{d}t = f(t_0)$

- Using the sifting property, we have $\mathcal{L}\left\{\delta(t - t_0)\right\} = e^{-st_0}$ and so $\mathcal{L}\left\{\delta(t)\right\} = 1$