Lecture 28, Nov 18, 2022

Differential Equations With Discontinuous Forcing Functions

- Example: flipping a switch, or turning a knob are all examples of discontinuous forcing functions
- Example 1: $y'' + 4y = g(t), y(0) = 0, y'(0) = 0, g(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \dfrac{t-5}{5} & 5 \leq t < 10 \\ 1 & t \geq 10 \end{cases}$
  - This forcing function is known as a *ramp*
  - First express $g(t)$ in terms of step functions: $g(t) = \dfrac{t-5}{5}u_5(t) - \dfrac{t-10}{5}u_{10}(t)$
  - $\mathcal{L}\{g\} = \dfrac{1}{5}\mathcal{L}\{(t-5)u_5(t) - (t-10)u_{10}(t)\} = \dfrac{1}{5s^2}(e^{-5s} - e^{-10s})$
  - $\mathcal{L}\{y'' + 4y\} = s^2Y(s) + 4Y(s) = \mathcal{L}\{g\}$
  - $Y(s) = (e^{-5s} - e^{-10s})\dfrac{1}{5s^2(s^2+4)}$
  - Let $H(s) = \dfrac{1}{s^2(s^2+4)}$, then $y(t) = \dfrac{u_5(t)h(t-5) - u_{10}(t)h(t-10)}{5}$ where $h(t) = \mathcal{L}^{-1}\{H(s)\}$
  - By partial fractions $H(s) = \dfrac{\frac{1}{4}}{s^2} - \dfrac{\frac{1}{4}}{s^2+4} \implies h(t) = \dfrac{1}{4}t - \dfrac{1}{8}\sin(2t)$

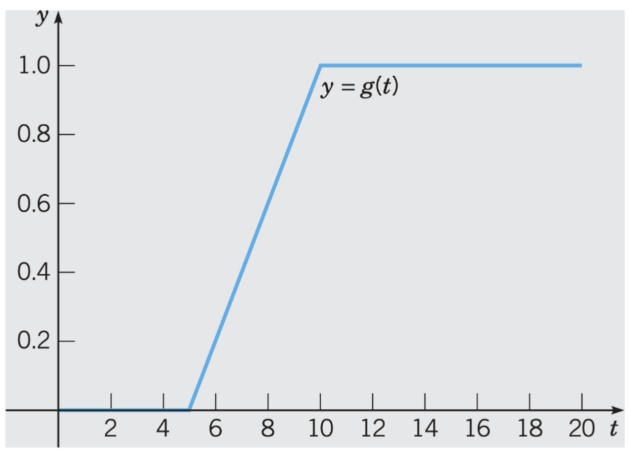

Figure 1: Ramp forcing function

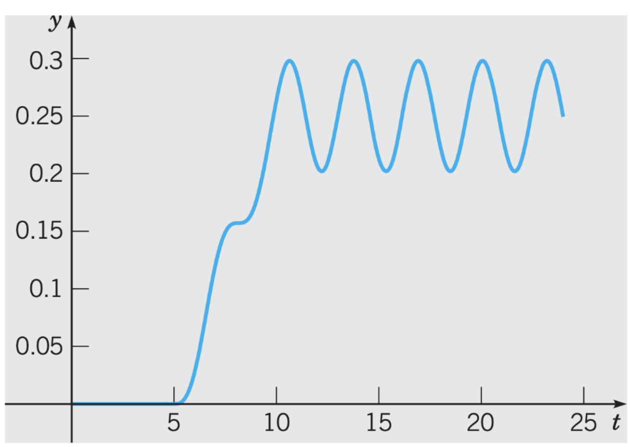

Figure 2: Example 1 solution

- Example 2: $y'' + \pi^2 y = f(t), y(0) = 0, y'(0) = 0$ where $f(t)$ is a square wave
  - Use the periodic function Laplace transform formula
  - We need a window function f_2 which we could construct as $f_2 = u_0(t) - u_1(t)$
  - $F_2(s) = \frac{1}{s}(1 - e^{-s})$
  - From the previous lecture $F(s) = \frac{F_2(s)}{1 - e^{-st}} = \frac{1 - e^{-s}}{s(1 - e^{-2s})} = \frac{1 - e^{-s}}{s(1 - e^{-2s})(1 + e^{-2s})} = \frac{1}{s(1 + e^{-s})}$
  - $\mathcal{L}\left\{y'' + \pi^2 y\right\} = (s^2 + \pi^2)Y(s) = F(s) \implies Y(s) = \frac{1}{s(1 + e^{-s})(s^2 + \pi^2)} = \frac{1}{s(s^2 + \pi^2)} \frac{1}{1 + e^{-s}}$
  - Let $H(s) = \frac{1}{s(s^2 + \pi^2)}$
  - $Y(s) = \sum_{k=1}^{\infty} (-1)^k e^{-ks} H(s)$
  - By partial fractions $h(t) = \frac{1}{\pi^2}(1 - \cos(\pi t))$
  - Therefore $y(t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{\pi^2}(1 - \cos(\pi(t - k)))u_k(t)$

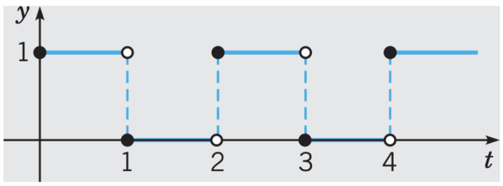

Figure 3: Square wave forcing function

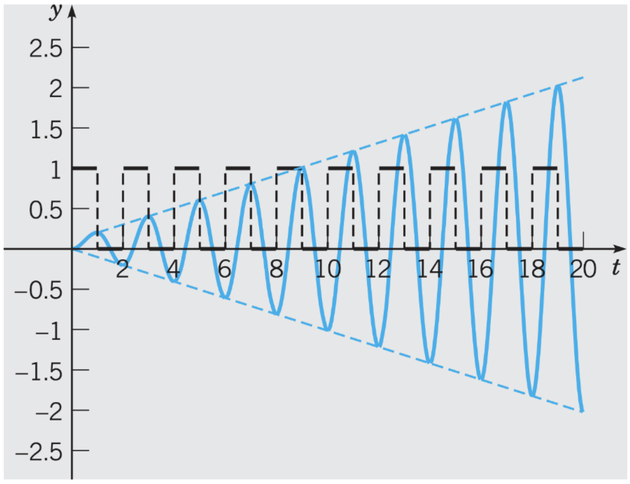

Figure 4: Example 2 solution