Lecture 27, Nov 17, 2022

Unit Step Function (Heaviside Function)

- Heaviside step function (aka indicator function): $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$
- Translated step function: $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$
- Indicator step function: $u_{cd}(t) = u_c(t) - u_d(t) = \begin{cases} 0 & t < c, t \geq d \\ 1 & c \leq t < d \end{cases}$
- From the step function we can construct other functions, e.g. a triangular pulse is $(-1+t)u_{12}(t) + (3-t)u_{23}(t)$

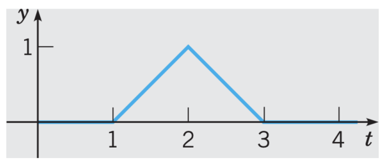

Figure 1: Triangular pulse

Laplace Transform of the Step Function

- If $\mathcal{L}\{f(t)\} = F(s), s > a \geq 0$, then $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s), s > a$
  - This is the dual of $\mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-c)$
- An exponential in the time domain is a shift in the s domain; an exponential in the s domain is also a shift in the time domain
- $\mathcal{L}\{u(t)\} = \dfrac{1}{s}, s > 0$
- $\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}, s > 0$
- $\mathcal{L}\{u_{cd}(t)\} = \mathcal{L}\{u_c(t)\} - \mathcal{L}\{u_d(t)\} = \dfrac{e^{-cs} - e^{-ds}}{s}, s > 0$

Periodic Functions

Definition

A function f is periodic if

$$f(t+T) = f(t)$$

where T is the period

- The window function: $f_T(t) = f(t)(1 - u_T(t)) = \begin{cases} f(t) & t \leq T \\ 0 & \text{otherwise} \end{cases}$
- We can use it to construct periodic functions as $f(t) = \displaystyle\sum_{n=0}^{\infty} f_T(t - nT)u_{nT}(t)$
- Using this, we can Laplace transform any periodic function

- $$\begin{aligned}\mathcal{L}\left\{f(t)\right\} &= \sum_{n=0}^{\infty} \mathcal{L}\left\{f_T(t-nT)u_{nT}(t)\right\} \\ &= \sum_{n=0}^{\infty} e^{-nTs}\mathcal{L}\left\{f_T(t)\right\} \\ &= \mathcal{L}\left\{f_T(t)\right\}\frac{1}{1-e^{-Ts}}\end{aligned}$$

Theorem

If f is periodic with period T and is piecewise continuous on $[0, T]$, then

$$\mathcal{L}\left\{f(t)\right\} = \frac{F_T(s)}{1-e^{-Ts}}$$

where $F_T(s) = \mathcal{L}\left\{f_T(t)\right\} = \mathcal{L}\left\{f(t)(1-u_T(t))\right\}$