Lecture 25, Nov 4, 2022

Properties of the Laplace Transform

1. Exponential in t is a shift in s: $\mathcal{L}\left\{f(t)\right\} = F(s), s > a \implies \mathcal{L}\left\{e^{ct} f(t)\right\} = F(s-c), s > a + c$
   - e.g. $\mathcal{L}\left\{e^{-2t}\sin(4t)\right\} = \dfrac{4}{(s+2)^2+16}, s > 2$
2. Derivative in t is a multiplication by s: $\mathcal{L}\left\{f'(s)\right\} = s\mathcal{L}\left\{f(t)\right\} - f(0)$
   - Derived by integration by parts
   - $\displaystyle \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$
     $= s\mathcal{L}\left\{f(t)\right\} - f(0)$
   - We need to assume that the function does not blow up
3. Corollary: $\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
   - e.g. $\dfrac{\mathrm{d}^4 y}{\mathrm{d}t^4} - y = 0$ with initial conditions $y(0) = 0$, $\left.\dfrac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=0} = 0$, $\left.\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right|_{t=0} = 0$, $\left.\dfrac{\mathrm{d}^3 y}{\mathrm{d}t^3}\right|_{t=0} = 1$
     - Take the Laplace transform of both sides
     - $s^4\mathcal{L}\left\{y(t)\right\} - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0) - \mathcal{L}\left\{y\right\} = 0$
     - $s^4\mathcal{L}\left\{y(t)\right\} - \mathcal{L}\left\{y\right\} = 1 \implies \mathcal{L}\left\{y(t)\right\} = \dfrac{1}{s^4-1}$
   - Starting with an ODE in t, we get a polynomial in s
4. Multiplication by t^n is an n-th derivative in s: $\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n F^{(n)}(s)$