Lecture 24, Nov 3, 2022

Definition of the Laplace Transform (again)

- So far the solutions to first order and second order ODEs all seem to contain exponentials and sinusoids
- Can we transform these solutions to something nicer?
- The Fourier transform: $\mathcal{F}\left\{f\right\}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,\mathrm{d}t$
  - This takes you from the time domain to the frequency domain
  - Can be though of as a dot product between the function of various sines and cosines of various frequencies
  - If we have a sine or a cosine, we get a very simple representation when taken to the frequency domain
  - However the Fourier transform doesn't work well with exponentials, which is why we need the Laplace transform
- The Laplace transform: $\mathcal{L}\left\{f\right\}(s) = \int_{0}^{\infty} e^{-st} f(t)\,\mathrm{d}t$
  - Note $\mathcal{L}\left\{f\right\}(\sigma + i\omega) = \int_{0}^{\infty} e^{-(\sigma+i\omega)t} f(t)\,\mathrm{d}t$
    * When $\sigma = 0$, we get the Fourier transform
  - The Fourier transform is a slice of the Laplace transform
  - The Fourier transform can only handle sines and cosines, which are pure oscillations that do not decay or grow, whereas with the Laplace transform we can also handle decaying exponentials
    * This makes it suitable for functions that appear in ODEs

Existence of the Laplace Transform

- $f(t)$ needs to be piecewise continuous for the integration to be possible
- $f(t)$ must not dominate the e^{-st}, otherwise the integral will diverge
- If $|f(t)| \leq Ke^{at}$ for $t < M$, then the Laplace transform exists if and only if $a < s$
  - We can show this using the limit comparison test

Theorem

The Laplace transform $\mathcal{L}\left\{f\right\}(s)$ exists for $s > a$ if:

1. f is piecewise continous on $0 \leq t \leq A$ for any positive A
2. f is of exponential order so that $|f(t)| \leq Ke^{at}$ when $t \geq M$

- Proof: $\mathcal{L}\left\{f\right\}(s) = \int_{0}^{\infty} e^{-st} f(t)\,\mathrm{d}t = \int_{0}^{M} e^{-st} f(t)\,\mathrm{d}t + \int_{M}^{\infty} e^{-st} f(t)\,\mathrm{d}t$
  - The first part exists by hypothesis 1, the second exists by hypothesis 2