Lecture 23, Oct 31, 2022

The Laplace Transform

Definition

The Laplace transform of a function $f(t)$ is

$$F(s) = \mathcal{L}\left\{f\right\}(s) = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t$$

- The Laplace transform is analogous to a change of coordinates in linear algebra
  - We're taking a function $f(t)$ to get back another function $F(s)$
  - This integral of the product of functions is akin to a dot product, but for functions; e^{-st} is a basis
    * We like a basis of e^{-st} because its derivative is proportional to itself

Example Transforms

- $\begin{aligned}\mathcal{L}\left\{1\right\} &= \int_0^\infty e^{-st}\, \mathrm{d}t \\ &= \left[-\frac{1}{s}e^{-st}\right]_0^\infty \\ &= \lim_{A\to\infty}\left(-\frac{e^{-sA}}{s} + \frac{1}{s}\right) \\ &= \frac{1}{s}, s > 0\end{aligned}$
- $\begin{aligned}\mathcal{L}\left\{e^{(a+ib)t}\right\} &= \int_0^\infty e^{-st}e^{(a+ib)t}\, \mathrm{d}t \\ &= \left[\frac{1}{a-s+ib}e^{((a-s)+ib)t}\right]_0^\infty \\ &= \lim_{A\to\infty}\left(\frac{e^{((a-s)+ib)A}}{a-s+ib} - \frac{1}{a-s+ib}\right) \\ &= \frac{1}{s-(a+ib)}, s > a\end{aligned}$
- $\begin{aligned}\mathcal{L}\left\{\sin t\right\} &= \mathcal{L}\left\{\frac{e^{it}-e^{-it}}{2i}\right\} \\ &= \frac{1}{2i}\mathcal{L}\left\{e^{it}\right\} - \frac{1}{2i}\mathcal{L}\left\{e^{-it}\right\} \\ &= -\frac{1}{2i}\frac{1}{-s+i} + \frac{1}{2i}\frac{1}{-s-i} \\ &= -\frac{1}{2i}\frac{-s-i-(-s+i)}{(-s)^2-i^2} \\ &= -\frac{1}{2i}\frac{-2i}{s^2+1} \\ &= \frac{1}{s^2+1}\end{aligned}$
- In reality we just look these up from a table

Linearity of the Laplace Transform

Theorem

The Laplace transform is linear:

$$\mathcal{L}\left\{c_1 f_1(t) + c_2 f_2(t)\right\} = c_1 \mathcal{L}\left\{f_1(t)\right\} + c_2 \mathcal{L}\left\{f_2(t)\right\}$$

If $\mathcal{L}\left\{f_1\right\}$ exists for $t > s_1$ and $\mathcal{L}\left\{f_2\right\}$ exists for $t > s_2$ then the linear combination exists for $t > \max(s_1, s_2)$