Lecture 22, Oct 28, 2022

Method of Variation of Parameters

- Consider a nonhomogeneous system: $\vec{\boldsymbol{x}}' = \boldsymbol{P}(t)\vec{\boldsymbol{x}} + \vec{\boldsymbol{g}}(t)$
  - We solve the homogeneous system $\vec{\boldsymbol{x}}' = \boldsymbol{P}(t)\vec{\boldsymbol{x}}$
  - To do this we need a fundamental set $\{\, \vec{\boldsymbol{x}}_1(t), \vec{\boldsymbol{x}}_2(t) \,\}$
    - This means $\vec{\boldsymbol{x}}_1' = \boldsymbol{P}(t)\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2' = \boldsymbol{P}(t)\vec{\boldsymbol{x}}_2$
  - We can write $\boldsymbol{X}' = \boldsymbol{P}(t)\boldsymbol{X}$, where $\boldsymbol{X} = \begin{bmatrix}\vec{\boldsymbol{x}}_1 & \vec{\boldsymbol{x}}_2\end{bmatrix}$
  - From this we can construct a general solution of the homogeneous system: $\vec{\boldsymbol{x}} = c_1\vec{\boldsymbol{x}}_1(t) + c_2\vec{\boldsymbol{x}}_2(t)$
- Guess solution: $u_1(t)\vec{\boldsymbol{x}}_1(t) + u_2(t)\vec{\boldsymbol{x}}_2(t) = \boldsymbol{X}(t)\vec{\boldsymbol{u}}(t)$ where $\vec{\boldsymbol{u}} = \begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}$
- Substitute into nonhomogeneous equation: $(\boldsymbol{X}(t)\vec{\boldsymbol{u}}(t))' = \boldsymbol{X}'(t)\vec{\boldsymbol{u}}(t) + \boldsymbol{X}(t)\vec{\boldsymbol{u}}'(t) = \boldsymbol{P}(t)\boldsymbol{X}(t)\vec{\boldsymbol{u}}(t) + \vec{\boldsymbol{g}}(t)$
  - Since $\boldsymbol{X}'(t) = \boldsymbol{P}(t)\boldsymbol{X}(t)$ this simplifies to just $\boldsymbol{X}(t)\vec{\boldsymbol{u}}'(t) = \vec{\boldsymbol{g}}(t)$
- Therefore $\vec{\boldsymbol{u}}'(t) = \boldsymbol{X}(t)^{-1}\vec{\boldsymbol{g}}(t)$
  - We know $\boldsymbol{X}(t)$ is invertible since the fundamental matrix always has a nonzero Wronskian
- Now we can integrate: $\vec{\boldsymbol{u}}(t) = \vec{\boldsymbol{c}} + \displaystyle\int \boldsymbol{X}(t)^{-1}\vec{\boldsymbol{g}}(t)\,\mathrm{d}t$
- $\vec{\boldsymbol{x}} = \boldsymbol{X}(t)\vec{\boldsymbol{u}}(t) = \boldsymbol{X}(t)\vec{\boldsymbol{c}} + \boldsymbol{X}(t)\displaystyle\int \boldsymbol{X}(t)^{-1}\vec{\boldsymbol{g}}(t)\,\mathrm{d}t$
  - Notice this consists of $\boldsymbol{X}(t)\vec{\boldsymbol{c}}$, which is the general solution to the homogeneous equation, plus a particular solution to the nonhomogeneous equation
- Note: For a 2x2 problem, $\boldsymbol{X}(t)^{-1} = \dfrac{1}{W[\vec{\boldsymbol{x}}_1(t), \vec{\boldsymbol{x}}_2(t)]}\begin{bmatrix}x_{22}(t) & -x_{12}(t)\\ -x_{21}(t) & x_{11}(t)\end{bmatrix}$

Theorem

The general linear nonhomogeneous system

$$\vec{\boldsymbol{x}}' = \boldsymbol{P}(t)\vec{\boldsymbol{x}} + \vec{\boldsymbol{g}}(t)$$

is solved by

$$\vec{\boldsymbol{x}} = \boldsymbol{X}(t)\vec{\boldsymbol{u}}(t) = \boldsymbol{X}(t)\vec{\boldsymbol{c}} + \boldsymbol{X}(t)\int \boldsymbol{X}(t)^{-1}\vec{\boldsymbol{g}}(t)\,\mathrm{d}t$$

where $\boldsymbol{X}(t)$ is the fundamental matrix of the system, and $\vec{\boldsymbol{c}}$ is a vector of constants determined by initial conditions

Second Order Nonhomogeneous ODE

- Consider the ODE $y'' + p(t)y' + q(t)y = g(t)$
  - We can use variation of parameters to solve this
- Convert to system: $\vec{\boldsymbol{x}} = \begin{bmatrix}0 & 1\\ -q(t) & -p(t)\end{bmatrix}\vec{\boldsymbol{x}} + \begin{bmatrix}0\\ g(t)\end{bmatrix}$
- Notice the fundamental matrix has the structure $\boldsymbol{X}(t) = \begin{bmatrix}y_1(t) & y_2(t)\\ y_1'(t) & y_2'(t)\end{bmatrix}$
- The particular solution:
$$\begin{aligned}\boldsymbol{X}(t)\int \boldsymbol{X}(t)^{-1}\vec{\boldsymbol{g}}(t)\,\mathrm{d}t &= \frac{1}{W[\vec{\boldsymbol{y}}_1(t), \vec{\boldsymbol{y}}_2(t)]}\begin{bmatrix}y_2'(t) & -y_2(t)\\ -y_1'(t) & y_1(t)\end{bmatrix}\begin{bmatrix}0\\ g(t)\end{bmatrix}\,\mathrm{d}t\\ &= \boldsymbol{X}(t)\int \frac{1}{W[\vec{\boldsymbol{y}}_1(t), \vec{\boldsymbol{y}}_2(t)]}\begin{bmatrix}-g(t)y_2(t)\\ g(t)y_1(t)\end{bmatrix}\,\mathrm{d}t\\ &= -\begin{bmatrix}y_1(t)\\ y_1'(t)\end{bmatrix}\int \frac{1}{W}g(t)y_2(t)\,\mathrm{d}t + \begin{bmatrix}y_2(t)\\ y_2'(t)\end{bmatrix}\int \frac{1}{W}g(t)y_1(t)\,\mathrm{d}t\end{aligned}$$
- We can now extract a particular solution for y: $y_p = -y_1(t)\displaystyle\int \frac{1}{W}g(t)y_2(t)\,\mathrm{d}t + y_2(t)\int \frac{1}{W}g(t)y_1(t)\,\mathrm{d}t$

Theorem

A particular solution to the general second order linear nonhomogeneous ODE

$$y'' + p(t)y' + q(t)y = g(t)$$

is

$$y_p = -y_1(t) \int \frac{1}{W} g(t) y_2(t)\,\mathrm{d}t + y_2(t) \int \frac{1}{W} g(t) y_1(t)\,\mathrm{d}t$$

where $\{ y_1, y_2 \}$ is the fundamental set of the homogeneous solution and

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Example

- $y'' + 4y = \dfrac{3}{\sin t}$
- Homogeneous solution:
  - $\lambda = \pm 2i$
  - Use the formula for the complex case: $y = c_1 e^{\mu t} \cos(\nu t) + c_2 e^{\mu t} \sin(\nu t)$
  - $y_1 = \cos(2t), y_2 = \sin(2t)$
- By variation of parameters, $y = c_1 \cos(2t) + c_2 \sin(2t) - \dfrac{\cos(2t)}{2} \displaystyle\int \frac{3}{\sin t} \sin(2t)\,\mathrm{d}t + \dfrac{\sin(2t)}{2} \displaystyle\int \frac{3}{\sin t} \cos(2t)\,\mathrm{d}t$