Lecture 21, Oct 27, 2022

Second Order Linear Nonhomogeneous ODEs

- Recall: All solutions to $\boldsymbol{Ax} = \boldsymbol{b}$ can be constructed by taking a particular solution to it and adding a solution to $\boldsymbol{Ax} = \boldsymbol{0}$
- Consider a nonhomogeneous second order linear ODE $ay'' + by' + cy = g(t)$ and its homogeneous counterpart $ay'' + by + cy = 0$
- 2 observations:
  1. Let y_h be a solution to the homogeneous ODE and y_p be a particular solution to the nonhomogeneous ODE, then $y_h + y_p$ solves the nonhomogeneous ODE
  2. Let $y_p, \hat{y}_p$ be two particular solutions to the nonhomogeneous ODE, then $\hat{y}_p - y_p = y_h$ solves the homogeneous ODE
- This means that given a nonhomogeneous ODE, we simply have to find a particular solution to it y_p, and the general solution y_h to the homogeneous ODE; then the general solution to the nonhomogeneous ODE is the sum of the two

Important

Given

$$ay'' + by' + cy = g(t)$$

the general solution can be found by

$$y_g = y_h + y_p$$

where y_h is the general solution to the homogeneous ODE $ay'' + by + cy = 0$ and y_p is one particular solution to the homogeneous ODE

Method of Undetermined Coefficients

- Involves guessing "trial solutions" based on the form of $g(t)$
  - For e^{rt} we guess Ae^{rt}
  - For $\sin(\omega t)$ or $\cos(\omega t)$ we guess $A\sin(\omega t) + B\cos(\omega t)$
  - For degree n polynomial we guess $B_0 + B_1 t + B_2 t^2 + \cdots + B_n t^n$
  - For a combination of these, we guess a combination of the corresponding guesses
  - If the guess solution appears in the homogeneous solution, multiply by t
- Example: $y'' - 3y - 4y = 3e^{2t}$
  - $\lambda = -1, 4$
  - General homogeneous solution: $y_h(t) = c_1 e^{-t} + c_2 e^{4t}$
  - Guess: $y = Ae^{2t}$
    * Plug this in we get $4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$
    * $-6A = 3 \implies A = -\frac{1}{2}$
  - Particular solution: $y_p = -\frac{1}{2}e^{2t}$
  - General solution: $y = -\frac{1}{2}e^{2t} + c_1 e^{-t} + c_2 e^{4t}$
- Example: $y'' - 3y' - 4y = 2e^{-t}$
  - General homogeneous solution: $y_h(t) = c_1 e^{-t} + c_2 e^{4t}$
  - Guess: $y = Ae^{-t}$
    * Plugging this in: $Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} = 2e^{-t}$
    * However this is equal to zero! This is because e^{-t} is already in our homogeneous solution
  - Guess: $y = Ate^{-t}$
    * $y' = Ae^{-t} - Ate^{-t}$
    * $y'' = -2Ae^{-t} + Ate^{-t}$
    * Plugging this in and solving we get $A = -\frac{2}{5}$

Example Problem: RLC Circuit

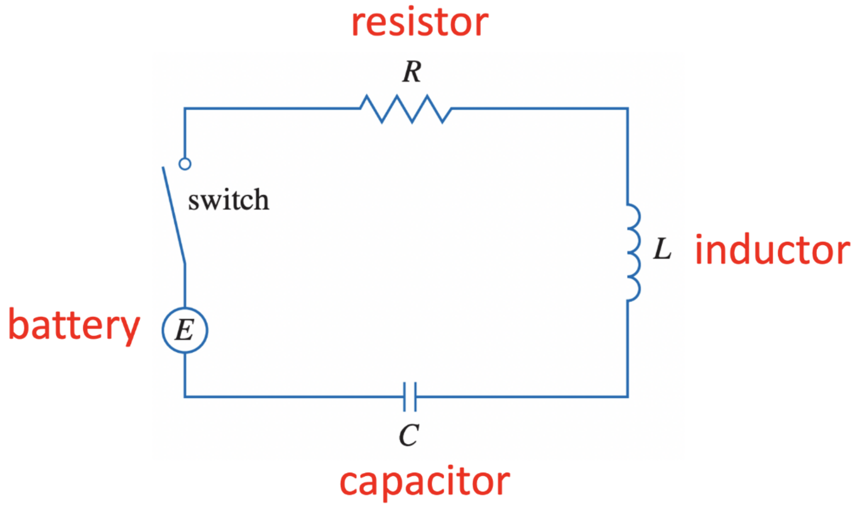

Figure 1: RLC Circuit

- Voltage across the inductor is $L\frac{\mathrm{d}I}{\mathrm{d}t}$; voltage across a capacitor is $\frac{Q}{C}$
- From Kirchhoff's Voltage Law: $L\frac{\mathrm{d}I}{\mathrm{d}t} + RI + \frac{Q}{C} = E(t)$
  - Since $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$ we can transform this into $L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = E(t)$
  - This is a second order linear nonhomogeneous ODE