Lecture 20, Oct 24, 2022

Second Order Linear Homogeneous Autonomous ODE

- Consider the ODE: $ay'' + by' + cy = 0$
- Recall we can express this as $\boldsymbol{x} = \begin{bmatrix} y \\ y' \end{bmatrix}, \boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \boldsymbol{x}$
  - The eigenvalues are: $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0 \implies a\lambda^2 + b\lambda + c = 0 \implies \lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The eigenvectors are $\boldsymbol{v} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$
- 3 cases for these eigenvalues:
  - Real and distinct (overdamped)
    * General solution $\boldsymbol{x} = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$, which gives $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ when λ are distinct
  - Real and equal (critically damped)
    * $\lambda_1 = \lambda_2 = -\dfrac{b}{2a}$
    * We only have a single eigenvector $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$, so we need to find the generalized eigenvector
    * $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{w} = \boldsymbol{v} \implies \boldsymbol{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
    * This gives us $\boldsymbol{x} = c_1 e^{\lambda_1 t} \boldsymbol{v}_1 + c_2 e^{\lambda_1 t}(t\boldsymbol{v}_1 + \boldsymbol{w}_1) = c_1 e^{\lambda t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda t} \begin{bmatrix} t \\ \lambda t + 1 \end{bmatrix}$
    * This gives $y = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$
  - Complex conjugates (underdamped)
    * $\lambda_1 = \mu + i\nu, \lambda_2 = \mu - i\nu$ where $\mu = -\dfrac{b}{2a}, \nu = \dfrac{\sqrt{4ac - b^2}}{2a}$
    * Construct the solution just like before and use Euler's identity: $\boldsymbol{x} = c_1 e^{(\mu + i\nu)t} \begin{bmatrix} 1 \\ \mu + i\nu \end{bmatrix} + c_2 e^{(\mu - i\nu)t} \begin{bmatrix} 1 \\ \mu - i\nu \end{bmatrix} = c_1 e^{\mu t} \begin{bmatrix} \cos \nu t \\ \mu \cos \nu t - \nu \sin \nu t \end{bmatrix} + c_2 e^{\mu t} \begin{bmatrix} \sin \nu t \\ \mu \sin \nu t + \nu \cos \nu t \end{bmatrix}$
    * This gives: $y = c_1 e^{\mu t} \cos \nu t + c_2 e^{\mu t} \sin \nu t$
- Example: $y'' + 5y' + 6y = 0$
  - Eigenvalues are $\lambda_1 = -2, \lambda_2 = -3$, real and distinct
  - Eigenvectors are then $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$
- Example: $y'' + y' + y = 0$
  - Eigenvalues are complex conjugates: $\lambda = -\dfrac{1}{2} \pm i\sqrt{\dfrac{3}{2}}$
- Example: $y'' - y' + \dfrac{1}{4}y = 0$
  - Eigenvalues are equal: $\lambda_1 = \lambda_2 = \dfrac{1}{2}$

Summary

The ODE $ay'' + by' + cy = 0$ is solved by:

1. When λ are real and distinct:
$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
2. When λ are complex conjugates:
$$y = c_1 e^{\mu t} \cos \nu t + c_2 e^{\mu t} \sin \nu t$$
3. When λs are real and equal:
$$y = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$$

where
$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
and $\lambda = \mu \pm i\nu$ when λ are complex