Lecture 19, Oct 21, 2022

Second Order Linear Differential Equations

- A second order linear ODE can be expressed as $y'' = f(t, y, y')$
  - Initial conditions $y(t_0) = y_0, y'(t_0) = y_1$
  - Notice two initial conditions are needed because we have 2 integration constants
- A second order linear ODE can be expressed as $y'' + p(t)y' + q(t)y = g(t)$
- A second order ODE can be written in terms of two first order ODEs:
  - Define $x_1 = y, x_2 = y'$
  - $y'' = x_2' = f(t, x_1, x_2)$
  - $y' = x_2 = x_1' \implies x_1' = x_2$
- If we had a second order linear ODE $y'' + p(t)y' + q(t)y = g(t)$, we can write it as a system of linear ODEs:
  - $x_1' = x_2, x_2' = -q(t)x_1 - p(x)x_2 + g(t)$
  - $\vec{\boldsymbol{x}}' = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \vec{\boldsymbol{x}} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$
  - Initial conditions can be added as $\vec{\boldsymbol{x}}(t_0) = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$

Theorem

Existence and Uniqueness Theorem: Given

$$y'' + p(t)y' + q(t)y = g(t)$$

if $p(t), q(t), g(t)$ are continuous over $t_0 \in (\alpha, \beta)$, then there exists a unique solution over (α, β)

- Second order linear ODEs also obey the theorem of superposition