Lecture 18, Oct 20, 2022

The Matrix Exponential

- If a scalar IVP $x' = ax, x(0) = x_0$ can be solved by $x = e^{at}x_0$, then can we solve $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0$ with $\boldsymbol{x} = e^{\boldsymbol{A}t}\boldsymbol{x}_0$?
- We can define the matrix exponential $e^{\boldsymbol{A}t}$ using a Taylor series, similar to a scalar exponential

Definition

The matrix exponential

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2t^2 + \frac{1}{3!}\boldsymbol{A}^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k t^k}{k!}$$

- The matrix exponential has the same properties as the scalar exponential
  - $e^{\boldsymbol{0}t} = \boldsymbol{I}$
  - $\frac{\mathrm{d}}{\mathrm{d}t}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t}$
  - $e^{\boldsymbol{A}(t+\tau)} = e^{\boldsymbol{A}t}e^{\boldsymbol{A}\tau}$
  - $(e^{\boldsymbol{A}t})^{-1} = e^{-\boldsymbol{A}t}$
  - $Ae^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}$
  - $e^{(\boldsymbol{A}+\boldsymbol{B})t} = e^{\boldsymbol{A}t}e^{\boldsymbol{B}t}$, but only if $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$

Theorem

Given an ODE $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$,

$$\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t}$$

is a solution to this ODE, and satisfies $\boldsymbol{\Phi}(0) = \boldsymbol{I}$

- Note that this is a matrix differential equation; this contains multiple solutions $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots$, which forms the fundamental set
- The matrix exponential is also sometimes known as the *special fundamental matrix*, because its columns are solutions that form a basis for the solution space
  - The general solution $\boldsymbol{x}(t)$ can be written as $\boldsymbol{x}(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) + \cdots = \boldsymbol{\Phi}(t)\boldsymbol{c}$
- If we are given an initial condition $\boldsymbol{X}(0) = \boldsymbol{x}_0$, then $\boldsymbol{x}(0) = \boldsymbol{\Phi}(0)\boldsymbol{c} \implies \boldsymbol{c} = \boldsymbol{\Phi}(0)^{-1}$; since $\boldsymbol{\Phi}(0) = \boldsymbol{I}$, the IVP is satisfied by $\boldsymbol{X}(t) = \boldsymbol{x}_0\boldsymbol{\Phi}(t)$

Theorem

The IVP

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0$$

is satisfied by

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0$$

Calculating The Matrix Exponential

- How do we take all those higher powers of $\boldsymbol{A}$?
- Eigendecomposition: $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}$ where $\boldsymbol{D}$ is a diagonal matrix of the eigenvalues and $\boldsymbol{V}$ is a matrix of all the eigenvectors
  - This works because $\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{D} = \boldsymbol{D}\boldsymbol{V}$
- Using eigendecomposition we can easily take higher powers: $\boldsymbol{A}^k = \boldsymbol{V}\boldsymbol{D}^k\boldsymbol{V}^{-1}$
  - This is great because $\boldsymbol{D}$ is diagonal, so $\boldsymbol{D}^k$ simply has the diagonal entries to the power of k

– This allows us to write $e^{\boldsymbol{A}t} = \boldsymbol{V}e^{\boldsymbol{D}t}\boldsymbol{V}^{-1}$, and $e^{\boldsymbol{D}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots \\ 0 & e^{\lambda_2 t} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$

Important

The IVP

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0$$

is satisfied by

$$\boldsymbol{x} = e^{\boldsymbol{A}t}\boldsymbol{x}_0 = \boldsymbol{V}e^{\boldsymbol{D}t}\boldsymbol{V}^{-1}\boldsymbol{x}_0$$