Lecture 17, Oct 17, 2022

First Order Linear Systems of Dimension n

- A general first order linear system can be described by $\vec{\boldsymbol{x}}' = \boldsymbol{P}(t) + \vec{\boldsymbol{g}}(t)$

Theorem

Given

$$\vec{\boldsymbol{x}}' = \boldsymbol{P}(t) + \vec{\boldsymbol{g}}(t), \vec{\boldsymbol{x}}(t_0) = \vec{\boldsymbol{y}_0}$$

and $\boldsymbol{P}(t), \boldsymbol{g}\vec{(t)}$ continuous over $t_0 \in (\alpha, \beta)$, then there exists a unique solution in the interval (α, β)

- Note $\boldsymbol{P} : \mathbb{R} \mapsto {}^n\mathbb{R}^n$ and $\vec{\boldsymbol{g}} : \mathbb{R} \mapsto \mathbb{R}^n$
- We can always centre the problem, so from now we assume $\vec{\boldsymbol{g}}(t) = 0$

Theorem

Principle of Superposition: Given

$$\vec{\boldsymbol{x}}' = \boldsymbol{P}(t)\vec{\boldsymbol{x}}$$

and $\vec{\boldsymbol{x}}_1(t), \vec{\boldsymbol{x}}_2(t), \cdots, \vec{\boldsymbol{x}}_n(t)$ are solutions, then

$$c_1\vec{\boldsymbol{x}}_1(t) + c_2\vec{\boldsymbol{x}}_2(t) + \cdots + c_n\vec{\boldsymbol{x}}_n(t)$$

is also a solution for any $c_1, c_2, \cdots, c_n$

This makes the set of all solutions a vector space

Definition

Functions $\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \cdots, \vec{\boldsymbol{x}}_n$ are linearly independent if

$$c_1\vec{\boldsymbol{x}}_1(t) + c_2\vec{\boldsymbol{x}}_2(t) + \cdots + c_n\vec{\boldsymbol{x}}_n(t) = \vec{0} \implies c_1 = c_2 = \cdots = c_n = 0$$

Definition

The Wronskian

$$W[\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \cdots, \vec{\boldsymbol{x}}_n](t) = \det \begin{bmatrix} | & | & & | \\ \vec{\boldsymbol{x}}_1 & \vec{\boldsymbol{x}}_2 & \cdots & \vec{\boldsymbol{x}}_n \\ | & | & & | \end{bmatrix} = \det(\boldsymbol{X}(t))$$

If $\det(\boldsymbol{X}(t)) \neq 0$ for all t, then the solutions are independent

Theorem

If $\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \cdots, \vec{\boldsymbol{x}}_n$ are linearly independent solutions, then any solution can be expressed as

$$\vec{\boldsymbol{x}}(t) = c_1\vec{\boldsymbol{x}}_1(t) + c_2\vec{\boldsymbol{x}}_2(t) + \cdots + c_n\vec{\boldsymbol{x}}_n(t)$$

for some set of unique $c_1, c_2, \cdots, c_n$

Such a set of $\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \cdots, \vec{\boldsymbol{x}}_n$ is known as the *fundamental set* of the ODE

- This set of $c_1, c_2, \cdots, c_n$ depends on the initial conditions: If $\vec{\boldsymbol{x}}(0) = \vec{\boldsymbol{b}}$, then $\vec{\boldsymbol{c}} = \boldsymbol{X}(0)^{-1}\vec{\boldsymbol{b}}$