Lecture 16, Oct 14, 2022

Numerical Integration Methods

- Riemann sums approximates the function as a series of constant value segments
- Trapezoidal rule approximates the function as a number of linear segments
- Simpson's one-third rule approximates the function as a series of parabolas
  - Take the current point, the next point, a point halfway, and fit a parabola
  - $\int_a^b f(x)\,\mathrm{d}x \approx \int a^b P(x)\,\mathrm{d}x = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$
    * This is just a closed-form solution for the integral of the parabola that passes through these 3 points

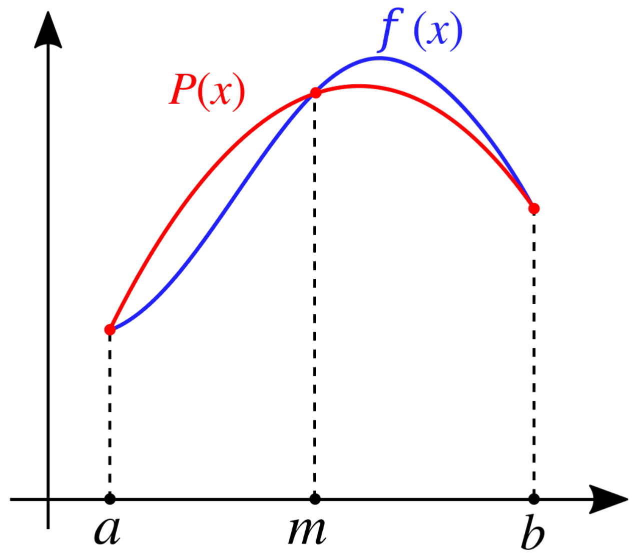

Figure 1: Simpson's Rule

Improved Euler Method

- When we solve an ODE, we are essentially integrating: $\phi(t_{n+1}) = \phi(t_n) + \int_{t_n}^{t_{n+1}} f(t, \phi(t))\,\mathrm{d}t$
- Euler's method, $y_{n+1} = y_n + hf(t_n, y_n)$ is essentially approximating f as a constant value $f(t, y) = f(t_n, y_n)$
  - This essentially makes a Riemann sum – so what if we used a trapezoidal sum instead?
- This leads to the improved Euler method $y_{n+1} = y_n + h\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}$
  - However, we can't quite use y_{n+1} in the right hand side because we haven't found it yet
  - We can use Euler's method to find an estimate for it

Definition

The Improved Euler/Heun/~~Abdullah~~ Method:

$$y_{n+1} = y_n + h\frac{f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))}{2}$$

- IEM is a *second order* method – local truncation error is $O(h^3)$ and global truncation error is $O(h^2)$
  - However, IEM requires two function evaluations per step

- But if $h \ll \frac{1}{2}$ this still makes IEM much more efficient

Runge-Kutta Method

Definition

The Runge-Kutta Method:

$$y_{n+1} = y_n + h\frac{s_{n1} + 2s_{n2} + 2s_{n3} + s_{n4}}{6}$$

where

$$\begin{aligned} s_{n1} &= f(t_n, y_n) \\ s_{n2} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hs_{n1}\right) \\ s_{n3} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hs_{n2}\right) \\ s_{n4} &= f(t_n + h, y_n + hs_{n3}) \end{aligned}$$

- The Runge-Kutta method essentially approximates with a parabola (Simpson's Rule)
- The derivative is evaluated at the current point, the next point, and also the point in the middle
- Runge-Kutta is a *fourth order* method – local truncation error is $O(h^5)$ and global truncation error is $O(h^4)$
  - Each step requires 4 function evaluation

Adaptive Step Sizes

- What if we could use smaller step sizes where it's needed?
- Run a standard step of Euler's method, and one IEM step; if we assume that the IEM gives the absolute truth, then the difference between the two approximations is the error
- The local truncation error should scale like h^2
  - $\frac{e^{\text{est}}_{n+1}}{h^2} = \frac{|y^{\text{euler}}_{n+1} - y^{\text{IEM}}_{n+1}|}{h^2} \approx \text{const}$
- If we adjust the step size to h_{new}, with some new local truncation error ϵ, then $\frac{\epsilon}{h^2_{\text{new}}} \approx \text{const}$
- Therefore if we want to keep the local truncation error ϵ constant, then we can have $\frac{e^{\text{est}}_{n+1}}{h^2} = \frac{\epsilon}{h^2_{\text{new}}}$

Important

To keep the error rougly fixed at ϵ, adjust the step size as

$$h_{\text{new}} = h\sqrt{\frac{\epsilon}{e^{\text{est}}_{n+1}}}$$

where $e^{\text{est}}_{n+1} = |y^{\text{euler}}_{n+1} - y^{\text{IEM}}_{n+1}|$