Lecture 15, Oct 13, 2022

Errors in Numerical Approximations

- Round-off errors
- Euler's method relies on successive linear approximations
- Global truncation error: $E_n = \phi(t_n) - y_n$, error accumulated across all steps
  - We use y_n instead of $\phi(t_n)$ to determine y_{n+1} so errors can accumulate
- Local truncation error: Error due to the linear approximation only

Local Truncation Error

- Consider a general ODE $y' = f(t, y)$ with solution $\phi(t)$, so $\phi'(t) = f(t, \phi(t))$
- With Euler's method, $y_{n+1} = y_n + hf(t_n, y_n)$
- The error is $|y_{n+1} - \phi(t_{n+1})|$
- Using a Taylor approximation: $\phi(t_{n+1}) = \phi(t_n) + \phi'(t_n)h + \frac{1}{2}\phi''(\bar{t}_n)h^2$ where $t_n < \bar{t}_n < t_n + h$
  - This is an exact equality due to $\bar{t}_n$ (Taylor's Remainder Theorem)
  - Note we assumed that ϕ is twice-differentiable and continuous in its derivatives
- The error is $|\phi(t_{n+1} - y_{n+1})| = (\phi(t_n) - y_n) + h(f(t_n, \phi(t_n)) - f(t_n, y_n)) + \frac{1}{2}\phi''(\bar{t}_n)h^2$
  - The term $\phi(t_n) - y_n = 0$ because this is a local error
  - Since $\phi(t_n) = y_n$ the middle term is also 0
  - Therefore the error is $\frac{1}{2}\phi''(\bar{t}_n)h^2$
- To bound $\frac{1}{2}\phi''(\bar{t}_n)h^2$, we assume $|\phi''(t)| \leq M$ so $|e_n| \leq \frac{Mh^2}{2}$

Global Truncation Error

- The number of steps is $n = \frac{T - t_0}{h}$
- The global truncation error can be approximated as $n\frac{Mh^2}{2} = \frac{(t - t_0)Mh}{2}$
- Notice the global truncation error decreases linearly with h
  - Euler's method is a first-order method because the power of h is 1

Assumptions

- For ϕ'' to be continuous to invoke the Taylor series, we need $\frac{\partial f}{\partial t}(t, \phi(t)) + \frac{\partial f}{\partial y}(t, \phi(t))f(t, \phi(t))$ continuous since $\phi' = f(t, \phi(t))$
  - This means we need f, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous