Lecture 12, Oct 3, 2022

Summary of Cases of Eigenvalues

- Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - The characteristic equation is $\lambda^2 - (a+d)\lambda + ad - bc = 0 \implies \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$
  - Therefore $\lambda = \dfrac{\operatorname{tr}(A) \pm \sqrt{(\operatorname{tr}(A))^2 - 4\det(A)}}{2}$
- Let $p = \operatorname{tr} A, q = \det A$, then the sign of $p^2 - 4q$ determines the behaviour of the ODE:
  - On the parabola, $p^2 = 4q$, we get two real, equal eigenvalues
  - Below the parabola $4q < p^2$, we get two real, distinct eigenvalues
  - Above the parabola $4q > p^2$, we get two complex eigenvalues that are complements
- Recall $\det A = \lambda_1\lambda_2$, so if $\det A < 0$ the eigenvalues have different signs; if $\det A > 0$ they have the same sign
  - Below the p axis the determinant is negative so the eigenvalues have different signs, so we get saddle points (semistable equilibria)
  - Above it we get either stable or unstable equilibrium since eigenvalues have the same sign
    * Between the p axis and parabola, on the right, the trace is positive, so both eigenvalues must be positive, leading to an unstable equilibrium
    * On the left the trace is negative, so both eigenvalues must be negative, leading to a stable equilibrium

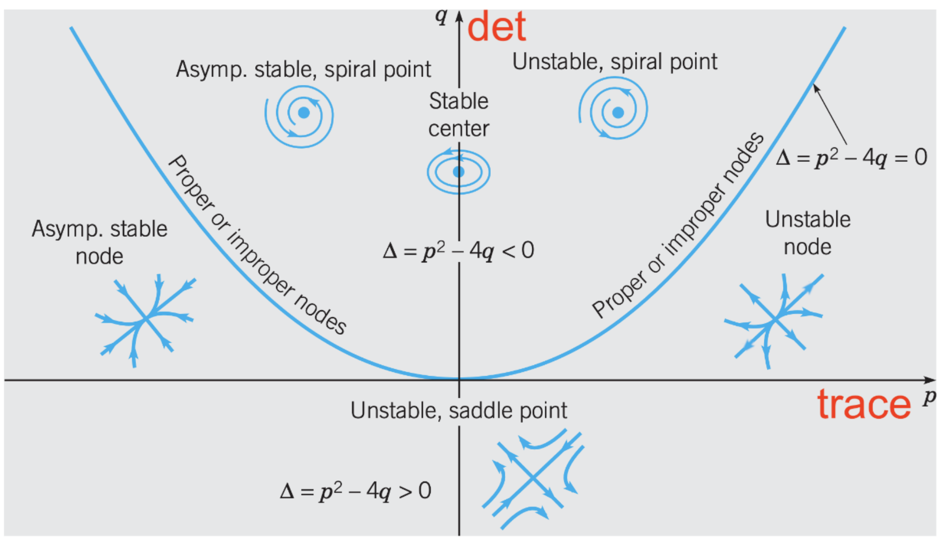

Figure 1: Summary of possible cases of the determinant and trace

Repeated Eigenvalues (and Eigenvectors)

- On the parabola we have repeated eigenvalues, e.g. $\boldsymbol{x}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{x}$
  - In this case we have repeated eigenvalues, but two distinct eigenvectors
- Another example: $\boldsymbol{x}' = \begin{bmatrix} m' \\ w' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} m \\ w \end{bmatrix}$
  - In this case we have $\lambda_1 = \lambda_2 = -\frac{1}{2}$, but only one eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  - Such matrices are *defective*; if we follow our usual procedure we only get one solution, which does

not span the full solution space

- Notice in this system w' does not depend on m, so we can solve it independently to get $w = c_2 e^{-\frac{t}{2}}$
  - Substituting this back in we get $m' = -\frac{1}{2}m + c_2 e^{-\frac{t}{2}}$ which is a FO linear ODE
  - Using integrating factors we get $m = c_2 t e^{-\frac{t}{2}} + c_1 e^{-\frac{t}{2}}$
  - The final solution is $\boldsymbol{x} = c_1 e^{-\frac{t}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left(t e^{-\frac{t}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-\frac{t}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$
    * But wait, where did $\boldsymbol{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ come from? What does it mean?
- Our solution has the form $\boldsymbol{x}_2 = t e^{-\frac{t}{2}} \boldsymbol{v} + t e^{-\frac{t}{2}} \boldsymbol{w}$
  - Substituting this in: $e^{-\frac{t}{2}} \boldsymbol{v} - \frac{1}{2} t e^{-\frac{t}{2}} \boldsymbol{v} - \frac{1}{2} e^{-\frac{t}{2}} \boldsymbol{w} = \boldsymbol{A} \left(t e^{-\frac{t}{2}} \boldsymbol{v} + t e^{-\frac{t}{2}} \boldsymbol{w} \right)$
  - Notice for this to hold we must have $-\frac{1}{2} t e^{-\frac{t}{2}} \boldsymbol{v} = \boldsymbol{A} t e^{-\frac{t}{2}}$ and $e^{-\frac{t}{2}} \boldsymbol{v} - \frac{1}{2} e^{-\frac{t}{2}} \boldsymbol{w} = \boldsymbol{A} e^{-\frac{t}{2}} \boldsymbol{w}$
  - This gives us $\left(\boldsymbol{A} + \frac{1}{2} \boldsymbol{I} \right) \boldsymbol{v} = 0$ and $\left(\boldsymbol{A} + \frac{1}{2} \boldsymbol{I} \right) \boldsymbol{w} = \boldsymbol{v}$
    * $\left(\boldsymbol{A} + \frac{1}{2} \boldsymbol{I} \right) \boldsymbol{w} = \boldsymbol{v}$ is a *generalized eigenvector equation*, where $\boldsymbol{w}$ is the *generalized eigenvector*
    * Solving this gives us $\boldsymbol{w} = \begin{bmatrix} k \\ 1 \end{bmatrix}$, so we can choose $k = 0$ and form our solution

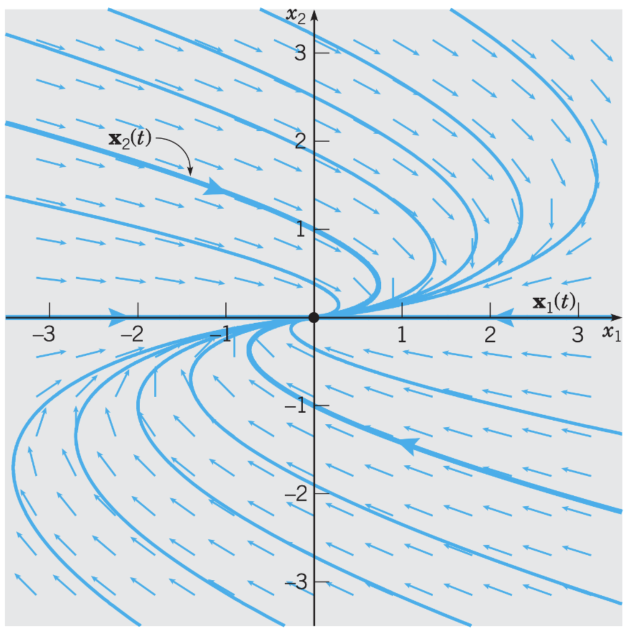

Figure 2: Solution to the system $\boldsymbol{x}' = \begin{bmatrix} m' \\ w' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} m \\ w \end{bmatrix}$

Definition

The generalized eigenvector is a vector that satisfies $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{w} = \boldsymbol{v}$, where $\boldsymbol{v}$ is the repeated eigenvector and λ is the repeated eigenvalue

Summary

When eigenvalues and eigenvectors are equal:

1. Write the first solution $\boldsymbol{x}_1(t) = e^{\lambda t}\boldsymbol{v}$ where $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v} = 0$
2. Write the second solution $\boldsymbol{x}_2(t) = te^{\lambda t}\boldsymbol{v} + e^{\lambda t}\boldsymbol{w}$ where $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{w} = \boldsymbol{v}$
3. The general solution is then $\boldsymbol{x} = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t)$

This works even when A is not triangular