Lecture 11, Sep 30, 2022

Lotka-Volterra (Predator-Prey)

- $\begin{cases} x' = \alpha x - \beta xy \\ y' = -\gamma y + \delta xy \end{cases}$
- Equilibrium point exists at $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{\delta} \\ \frac{\alpha}{\beta} \end{bmatrix}$

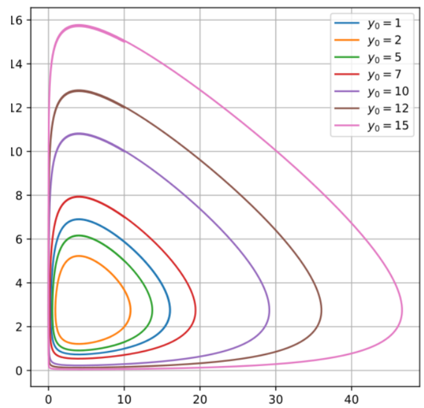

Figure 1: Solution curves to the system

- This system is nonlinear, so we have to linearize it
- We will linearize around the equilibrium
- The Jacobian evaluated is $J = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix}$
  - At equilibrium this is $\begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\gamma}{\beta} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- The eigenvalues of this system are complex! $\lambda = \pm\sqrt{\alpha\gamma}$
  - When eigenvalues are complex, solutions have spirals

Complex Eigenvalues

- Theorem: If A is a real matrix, then its eigenvalues come in complex conjugate pairs
  - Eigenvalues also come in complex conjugate pairs, e.g. if $v_1 = \begin{bmatrix} 1 - i5 \\ 2 + i \end{bmatrix}$ then $v_2 = \begin{bmatrix} 1 + i5 \\ 2 - i \end{bmatrix}$
- Suppose $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{A}$ has complex eigenvalues
  - If we follow our usual approach we would get $\boldsymbol{x}_1(t) = e^{(\mu+i\nu)t}\boldsymbol{v}_1, \boldsymbol{x}_2(t) = e^{(\mu-i\nu)t}\bar{\boldsymbol{v}}_1$, but these are not real solutions
- Let $\boldsymbol{v}_1 = \boldsymbol{a} + i\boldsymbol{b} \implies \boldsymbol{x}_1(t) = e^{\mu t}(\cos(\nu t) + i\sin(\nu t))(\boldsymbol{a} + i\boldsymbol{b})$

$$= e^{\mu t}(\boldsymbol{a}\cos\nu t - \boldsymbol{b}\sin\nu t) + ie^{\nu t}(\boldsymbol{a}\sin\nu t + \boldsymbol{b}\cos\nu t)$$
$$= \boldsymbol{u}(t) + i\boldsymbol{w}(t)$$

- $\boldsymbol{u}(t)$ and $\boldsymbol{w}(t)$ form the fundamental set of solutions: $\boldsymbol{x} = c_1\boldsymbol{u}(t) + c_2\boldsymbol{w}(t)$
  - To verify this, we need to verify that they're both solutions and the Wronskian is nonzero (for now we will take this as a given)

- Example: $\boldsymbol{x}' = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} \\ 2 & -\frac{1}{2} \end{bmatrix} \boldsymbol{x} \implies \lambda_1 = \frac{3i}{2}, \vec{v}_1 = \begin{bmatrix} 5 \\ 2-6i \end{bmatrix}, \lambda_2 = -\frac{3i}{2}, \vec{v}_2 = \begin{bmatrix} 5 \\ 2+6i \end{bmatrix}$
    - $\boldsymbol{a} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$
    - $\nu = \frac{3}{2}, \mu = 0$
    - $u(t) = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \cos\left(\frac{3}{2}t\right) - \begin{bmatrix} 0 \\ -6 \end{bmatrix} \sin\left(\frac{3}{2}t\right)$
    - As $t \to \infty$ the solutions go in a cycle
    - There is a stable equilibrium at 0
    - Distinct complex eigenvalues with zero real part creates perfectly cyclical solutions

Complex Eigenvalues Cases

- Zero real part: circular solution that go nowhere as $t \to \infty$
    - Stable equilibrium at the origin
- Negative real part: solution spirals towards the origin as $t \to \infty$
    - Stable equilibrium at the origin
- Positive real part: solution spirals outwards from the origin as $t \to \infty$
    - Unstable equilibrium at the origin