Lecture 10, Sep 29, 2022

Eigenvalues of Linear ODE Systems

- Example: $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \begin{bmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \vec{x}, \vec{x}_0 = \begin{bmatrix} -16 \\ 20 \end{bmatrix}$
  - Eigenvalues and eigenvectors: $\lambda_1 = -\frac{7}{4}, \vec{v}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}, \lambda_2 = -\frac{1}{8}, \vec{v}_2 = \frac{1}{2}$
  - General solution: $\vec{x}(t) = c_1 e^{-\frac{7}{4}} \begin{bmatrix} 6 \\ -1 \end{bmatrix} + c_2 e^{-\frac{1}{8}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
  - Solution to IVP: $x(0) = \begin{bmatrix} -16 \\ 20 \end{bmatrix} = c_1 \begin{bmatrix} 6 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \implies c_1 = -4, c_2 = 8$
  - Unstable equilibrium (sink) at $(0, 0)$; all solutions approach this as $t \to \infty$
- To visualize these in 2D, first plot the eigenvectors; use the sign of the eigenvalues to determine the directions of solutions along the eigenvectors
- For a given solution, it moves in the direction of the "dominant" eigenvector faster (the dominant eigenvector is the one with the greatest magnitude in eigenvalue)
- With two negative eigenvalues, all solutions tend towards the equilibrium at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$
  - The equilibrium at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a *sink* and stable equilibrium
- With two positive eigenvalues, all solutions (except for the one starting at the origin) diverge towards infinity
  - The equilibrium at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a *source* and unstable equilibrium
- With one positive and one negative eigenvalue, one of the eigenvectors is divergent and one is convergent
  - The equilibrium at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a semistable equilibrium
- With one negative and one zero eigenvalue, solutions converge towards the zero eigenvalue, which is an entire line of equilibrium points