Lecture 1, Sep 8, 2022

- Modelling temperature of a boba cup on a hot day: $u(t)$ where T_0 is the surrounding temperature
  - u is the dependent variable, t is the independent variable
- What is the problem with the following models?
  - $u' = u^2$
    - Temperature increases forever
  - $u' = u'' + 2u$
    - No dependence on the surrounding temperature
  - $u' = u - T_0$
    - u does not approach T_0
  - $u' = T_0 - u$
    - The environment is not taken into account (e.g. if the type of liquid changed, the equation can't account for it)
- Newton's Law of Cooling: The rate of change of temperature is negatively proportional to the difference between the temperature difference between the object and its surroundings
  - $u' = -k(u - T_0)$
    - k is the transmission coefficient

Note

Newton was an avid boba drinker[citation needed]

- Solution:
  - $$\begin{aligned} & \frac{\mathrm{d}u}{\mathrm{d}t} = -k(u - T_0) \\ \implies & \frac{\frac{\mathrm{d}u}{\mathrm{d}t}}{u - T_0} = -k \\ \implies & \frac{\mathrm{d}}{\mathrm{d}t} \ln|u - T_0| = -k \\ \implies & \ln|u - T_0| = -kt + C \\ \implies & u - T_0 = Ae^{-kt} \\ \implies & u = Ae^{-kt} + T_0 \end{aligned}$$
  - This gives us a family of curves, all with different initial conditions (*integral curves*)
  - In general we know $u(0)$ or $u(t_0)$ for some t_0 so we can solve for A

Lecture 2, Sep 9, 2022

Classification of Differential Equations

- Ordinary vs Partial Differential Equations
  - PDEs have partial derivatives, resulting from the presence of multiple independent variables
- Order
  - The highest derivative that appears in the equation
- Linear vs Nonlinear
  - The most general nth order ODE can be expressed as $F(t, y, y', \cdots, y^{(n)}) = 0$
  - A linear ODE can be written as $a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t)$
    - a_n can depend on t and t alone
  - The linear DE is homogeneous if $g(t) = 0$
- Autonomous vs Nonautonomous
  - An autonomous ODE does not explicitly depend on t, e.g. $y' = y$ is autonomous, $y' = ty$ is not
- Separable vs Nonseparable
  - A first order ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$ is separable if we can decompose $f(t, y) = p(t)q(y)$

- Example: $\frac{\mathrm{d}u}{\mathrm{d}t} = -k(u - T_0)$ is a first order, linear, nonhomogeneous, autonomous, separable ODE

Lotka-Volterra (Predator-Prey)

- Modelling the number of zombies in an apocalypse, where x is the number of people and y is the number of zombies, assumptions:
  1. Zombies eat people
     - $x' = -\beta xy$
     - The rate at which people get eaten is proportional to the number of zombies and people
  2. People reproduce
     - $x' = \alpha x$
  3. Zombies suffer natural death and emigration
     - $y' = \delta xy - \gamma y$
     - Zombies flourish when they're being fed; the more there are, the more are dying of natural causes
- This is summarized in the system: $\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -\gamma y + \delta xy \end{cases}$

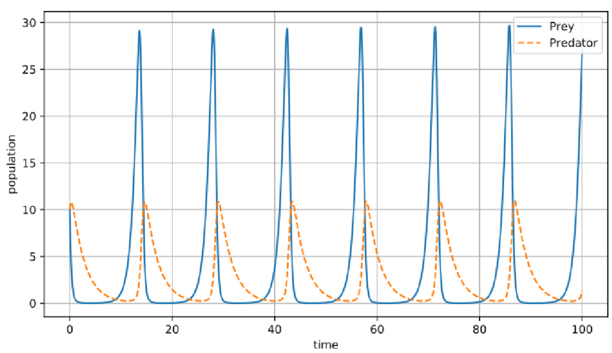

Figure 1: Cycle of predator-prey population

Lecture 3, Sep 12, 2022

Direction Fields

- Consider the DE $\frac{\mathrm{d}u}{\mathrm{d}t} = f(t, u)$
  - We can interpret this as the slope at each point is equal to some function of t and u
- We can draw a direction field, at each point (t, u) draw the slope $f(t, u)$
- Using a direction field, for any starting point we can follow it to trace out a solution to our ODE
- Direction fields allow us to visualize solutions to DEs without having to actually solve it

Equilibria

- Notice for this DE, all solutions tend towards the **equilibrium** $u = 60$
  - If we start from the equilibrium, we never move away from it, which lends to the definition:

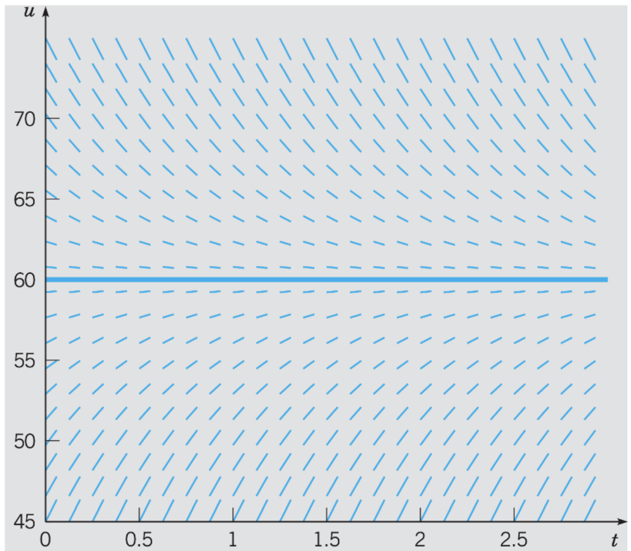

Figure 2: Direction field for $u' = -1.5(u - 60)$

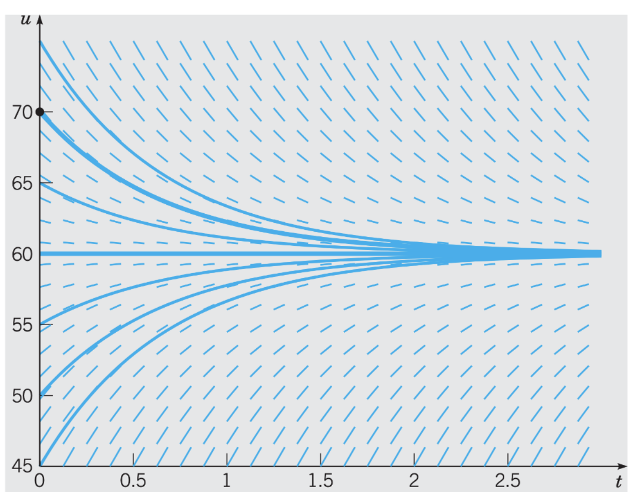

Figure 3: Direction field for $u' = -1.5(u - 60)$ with overlaid integral curves

Definition

Given a first order autonomous DE $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$, equilibrium solutions are those satisfying $f(y) = \frac{\mathrm{d}y}{\mathrm{d}t} = 0$

Equilibrium points are also known as critical points, fixed points, stationary points, etc

Definition

3 types of equilibria:

- Stable equilibrium: other solutions tend towards the equilibrium solution
- Unstable equilibrium: other solutions diverge from the equilibrium solution
- Semi-stable equilibrium: other solution tend towards the equilibrium on one side and diverge from it on the other

- For this DE, we have a stable equilibrium since all solutions approach the equilibrium solution $u = 60$
- $\frac{\mathrm{d}p}{\mathrm{d}t} = rp - a$ has an unstable equilibrium of $p = \frac{a}{r}$, since all other solutions diverge from this point

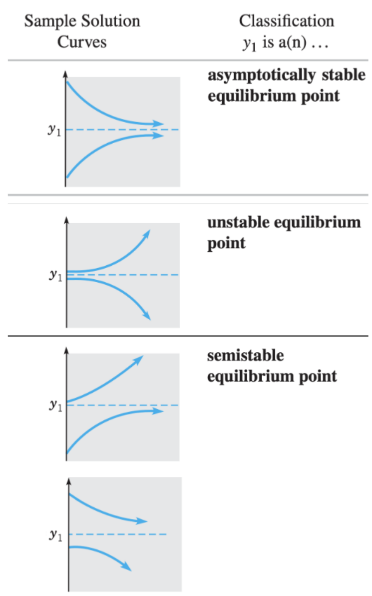

Figure 4: Types of eqilibrium

- Example: Find and classify equilibria of $y' = \cos y$:
  - $y' = 0 \implies \cos y = 0 \implies y = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$
  - The equilibrium at $\frac{\pi}{2}$ is stable, then $\frac{3\pi}{2}$ is unstable, $\frac{5\pi}{2}$ is stable, and so on
- On the plot, points where y' crosses from positive to negative are stable; points where y' crosses from negative to positive are unstable

Lecture 4, Sep 15, 2022

Method of Integrating Factors

- Consider a first order linear ODE: $\frac{\mathrm{d}u}{\mathrm{d}t} + p(t)u = g(t)$:

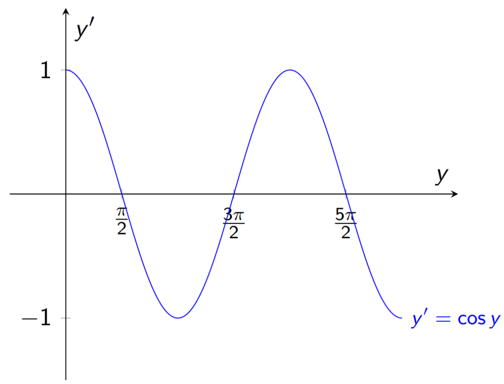

Figure 5: Plot of $y' = \cos y$

- Assume we have a function $\mu(t)$ such that $\mu(t)p(t) = \frac{\mathrm{d}\mu}{\mathrm{d}t}$
- Multiply by μ: $\mu(t)\frac{\mathrm{d}u}{\mathrm{d}t} + \mu(t)p(t)u = \mu(t)\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\mathrm{d}\mu}{\mathrm{d}t}u = \mu(t)g(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}(\mu u) = \mu(t)g(t) \implies u = \frac{\int \mu(t)g(t)\,\mathrm{d}t}{\mu(t)}$
- Choose μ: $\frac{\mathrm{d}\mu}{\mathrm{d}t} = \mu(t)p(t) \implies \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}t} = \mu(t)p(t) \implies \mu(t) = e^{\int p(t)\,\mathrm{d}t}$

Important

Method of integrating factors: The solution to $\frac{\mathrm{d}u}{\mathrm{d}t} + p(t)u = g(t)$ is $u(t) = \frac{1}{\mu}\left(\int \mu(t)g(t)\,\mathrm{d}t + C\right)$, where the integrating factor $\mu(t) = \exp\left(\int p(t)\,\mathrm{d}t\right)$

Example

- Example: $u' = -k(u - T_0 - A\sin(\omega t))$
  - The $A\sin(\omega T)$ term represents seasonal temperature variations
  - Put in standard form: $u' + ku = kT_0 + kA\sin(\omega t)$
  - Calculate integrating factor: $\mu = \exp\left(\int k\,\mathrm{d}t\right) = e^{kt}$
  - General solution:
$$\begin{aligned} u(t) &= \frac{1}{e^{kt}}\left(\int e^{ks}(kT_0 + kA\sin(\omega s))\,\mathrm{d}s + C\right) \\ &= T_0 + \frac{1}{e^{kt}}\left(\int e^{ks}kA\sin(\omega s)\,\mathrm{d}s\right) + C\frac{1}{e^{kt}} \\ &= \cdots \\ &= T_0 + \frac{kA}{k^2+\omega^2}(k\sin(\omega t) - \omega\cos(\omega t)) + C\frac{1}{e^{kt}} \end{aligned}$$
- Notice all solutions converge to a single solution $u(t) = T_0 + \frac{kA}{k^2+\omega^2}(k\sin(\omega t) - \omega\cos(\omega t))$
  - The dominant term is completely independent of initial condition C
- However, this is not an equilibrium because $\frac{\mathrm{d}u}{\mathrm{d}t} \neq 0$

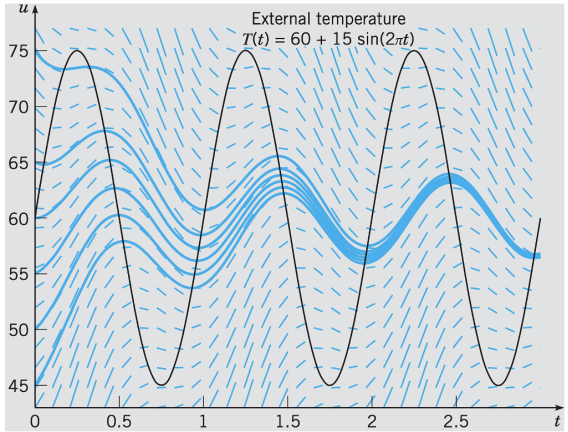

Figure 6: Solution curves

Lecture 5, Sep 16, 2022

Rocket Science

- Consider Earth with radius R, rocket with mass m at height x with velocity v, and gravitational acceleration g
- Given $ma = F, v(0) = v_0$ as the initial rocket velocity
  - The force of gravity is $\frac{mgR^2}{(R+x)^2}$, since for $x = 0 \implies \frac{mgR^2}{R^2} = mg$ is the gravitational force on the Earth's surface
  - This gives $ma = F = -\frac{mgR^2}{(R+x)^2}$ (since gravity works in the negative direction)
- The equation of motion is $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{gR^2}{(R+x)^2}$
  - Applying the chain rule: $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}v$
  - Using this we can eliminate t
  - Note by doing this we get a first order ODE, whereas making $a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ gives us a second order ODE
- $v\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{gR^2}{(R+x)^2}, v(0) = v_0$
  - Note here $v(0) = v_0$ means the velocity at position 0 instead of time 0
- Solution:
$$\begin{aligned} v\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{gR^2}{(R+x)^2} \\ \implies \frac{1}{2}v^2 &= \frac{gR^2}{R+x} + C \\ \implies v(x) &= \pm\sqrt{\frac{2gR^2}{R+x} + C} \end{aligned}$$
  - $v(0) = v_0 \implies \sqrt{2gR + C} = v_0 \implies C = v_0^2 - 2gR$
  - Final solution: $v = \pm\sqrt{\frac{2gR^2}{R+x} + v_0^2 - 2gR}$
- What is the maximum altitude x_{max} reached?
  - $v(x_{max}) = 0 \implies \frac{2gR^2}{R+x} + v_0^2 = 2gR \implies x_{max} = \frac{v_0^2 R}{2gR - v_0^2}$
- Given x_{max}, what v_0 do we need?
  - $v_0 = \sqrt{2gR\frac{x_{max}}{R + x_{max}}}$

- The escape velocity is then $\lim_{x_{max}\to\infty} v_0 = \sqrt{2gR}$

Lecture 6, Sep 19, 2022

Existence and Uniqueness of Solutions

- Given an ODE, does a solution exist, and is the solution unique?

Theorem

Given $y' + p(t)y = g(t), y(t_0) = y_0$, and p, g continuous over $t_0 \in (\alpha, \beta)$, then there exists a unique solution in the interval (α, β)

Theorem

Given $y' = f(t, y), y(t_0) = y_0$, and $f, \dfrac{\partial f}{\partial y}$ continuous over $(t_0, y_0) \in (\alpha, \beta) \times (\gamma, \delta)$, then there exists a unique solution in **some** interval $(t_0 - h, t_0 + h) \in (\alpha, \beta)$

The existence (but not uniqueness) of a solution can be established on the continuity of f alone

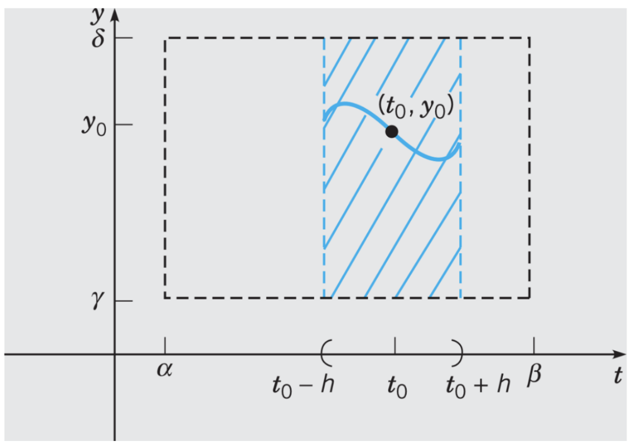

Figure 7: Visualization of the first-order nonlinear existence and uniqueness theorem

- The first-order nonlinear existence and uniqueness theorem only guarantees the existence and uniqueness of a solution in some interval within h, which we don't know
  - The linear version guarantees the entire continuous interval, whereas the nonlinear version only guarantees some smaller interval within the continuous interval
- Examples:
  - $y' + \dfrac{2}{t}y = 4t, y(1) = 2$
    * Use the linear theorem
    * p, g continuous except where $t = 0$
    * The initial condition lies within the continuous region, so a unique solution exists for $t \in (0, \infty)$
  - $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{3t^2 + 4t + 2}{2(y-1)}, y(0) = -1$
    * Use the nonlinear theorem
    * f is continuous except where $y = 1$
    * f_y is continuous except where $y = 1$

    * The initial condition lies in the region of continuity, so a unique solution exists for some region $t \in (-h, h)$
  - $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3t^2 + 4t + 2}{2(y-1)}, y(0) = 1$
    * Since f is not continuous at the initial condition, the theorem does not apply
    * Even though it's not guaranteed that a solution will exist, a solution may still exist
    * Solving the DE does yield a solution, but the solution is not unique

Important

Implication: The graphs of two solutions cannot intersect each other where the theorems hold (because this would violate the uniqueness of solutions)

"Just because you can't see a solution to an ODE doesn't mean you can't prove it exists." – Vardan, MAT292 (2022)

Lecture 7, Sep 22, 2022

Logistic Growth

- Simple growth model of $\frac{\mathrm{d}y}{\mathrm{d}t} = rt$ is unrealistic, as at some point the population needs to stop growing due to lack of resources
- Growth rate depending on population: $\frac{\mathrm{d}y}{\mathrm{d}t} = h(y)y$
  - Growth rate $h(y) = r - ay$
  - If y is small, then $h(y) > 0$ and the population grows
  - If y is large, then $h(y) < 0$ and the population dies off due to lack of resources
- This leads to the logistic equation (Verhulst equation): $\frac{\mathrm{d}y}{\mathrm{d}t} = (r - ay)y$
  - Equivalently $\frac{\mathrm{d}y}{\mathrm{d}t} = r\left(1 - \frac{y}{K}\right)y, K = \frac{r}{a}$
  - This is a first-order autonomous nonlinear ODE
- r is the intrinsic carrying capacity
- K is the saturation level, or environmental carrying capacity

Important

Logistic growth model:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r\left(1 - \frac{y}{K}\right)y$$

where $K = \frac{r}{a}$; solved by

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

assuming $y_0 < K$

- The line $y = K$ is a stable equilibrium
- $\frac{K}{2}$ is an inflection point, where the population curve goes from concave up to concave down
  - Rate of population growth begins to slow down
- The solution is $y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$, assuming $y_0 < K$

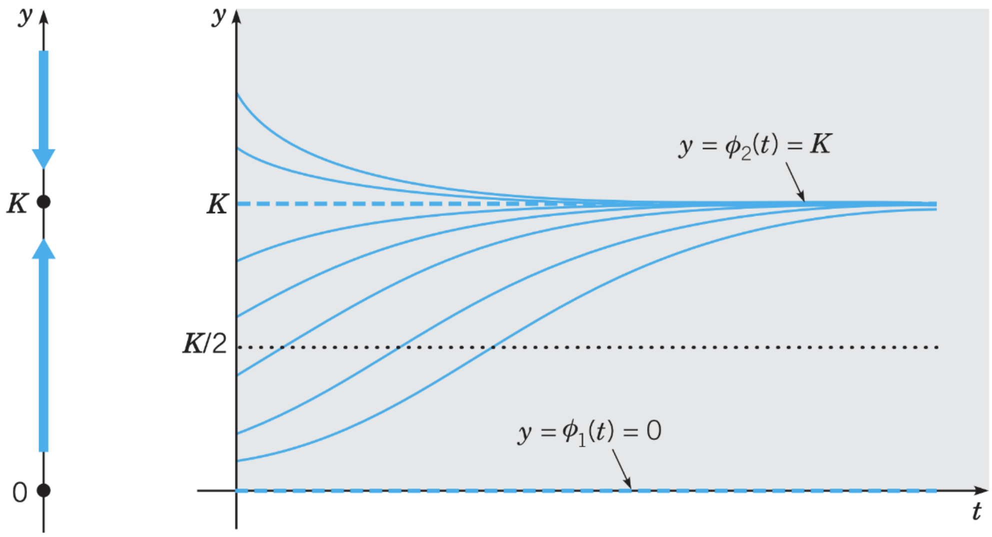

Figure 8: Solutions to the logistic model

Growth With a Threshold

- If the initial population is too low, they might all die out before the population can grow
- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -r\left(1 - \dfrac{y}{T}\right)y$

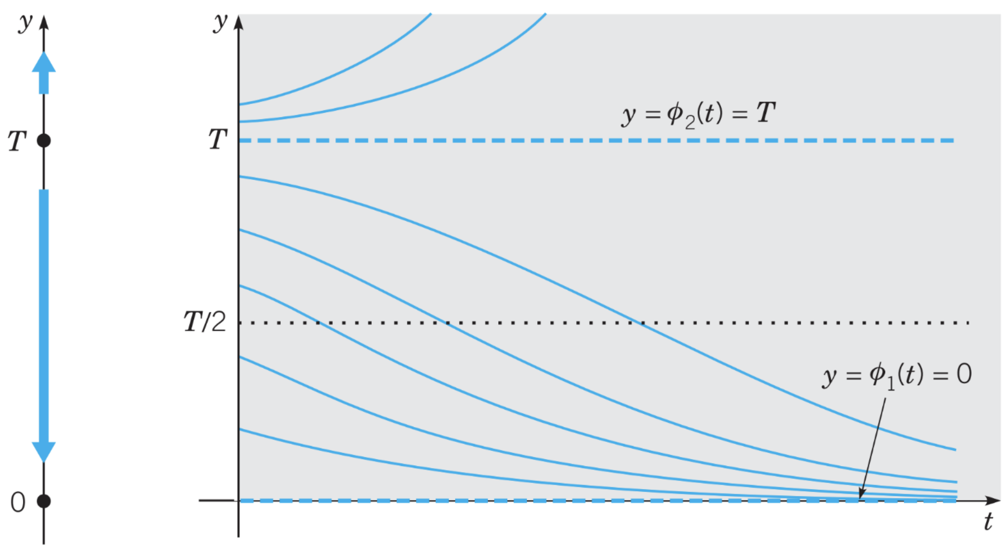

Figure 9: Solutions to the growth with a threshold model

- If $y_0 > T$ then the population keeps growing; if $y_0 < T$ then the population shrinks until everyone dies out
  - $y_0 = T$ is an unstable equilibrium
- Solution: $y = \dfrac{y_0 T}{y_0 + (T - y_0)e^{rt}}$

Logistic Growth With a Threshold

- Combine the two models: $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -r\left(1 - \dfrac{y}{T}\right)\left(1 - \dfrac{y}{K}\right)y$

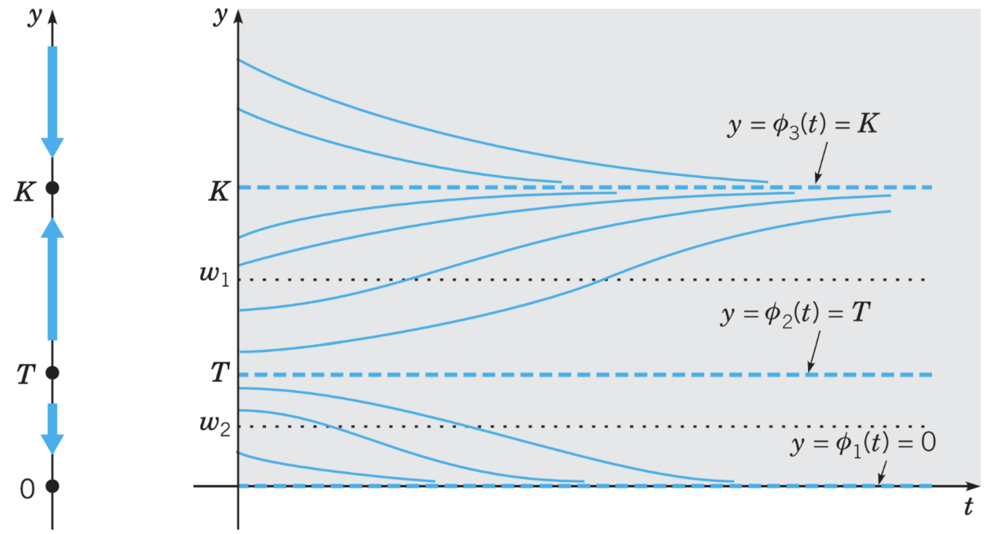

Figure 10: Solutions to the logistic growth with a threshold model

Lecture 8, Sep 23, 2022

Systems of Two First Order Linear ODEs

- Every week:
  - McDonald's gains 2 followers for every follower they have
  - Wendy's gains 3 followers for every follower they have
  - McDonald's loses 2 followers for every follower of Wendy's
  - Wendy's loses a follower for every follower of McDonald's
  - Both gain 300 followers
- $\begin{cases} m' = 2m - 2w + 300 \\ w' = 3w - m + 300 \end{cases}$
- As a matrix: $\begin{bmatrix} m' \\ w' \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} m \\ w \end{bmatrix} + \begin{bmatrix} 300 \\ 300 \end{bmatrix}$
- Notice the diagonal lines of vectors; they correspond to eigenvectors; the directions of arrows on the lines correspond to eigenvalues

First Order Linear ODEs of Dimension Two

- More generally, we have $\begin{bmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} p_{11}(t)x + p_{12}(t)y + g_1(t) \\ p_{21}(t)x + p_{22}(t)y + g_2(t) \end{bmatrix}$
  - This can be written as $\dfrac{\mathrm{d}\vec{z}}{\mathrm{d}t} = K(t)\vec{z} + \vec{g}(t)$
  - $\vec{z} = \begin{bmatrix} x \\ y \end{bmatrix}$ is the *state vector*, where x, y are *state variables*
  - $K(t) = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \vec{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$

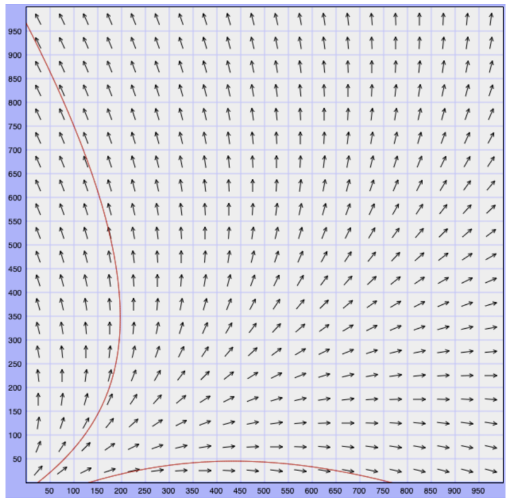

Figure 11: Direction field and trajectories/orbits in state space/phase plane

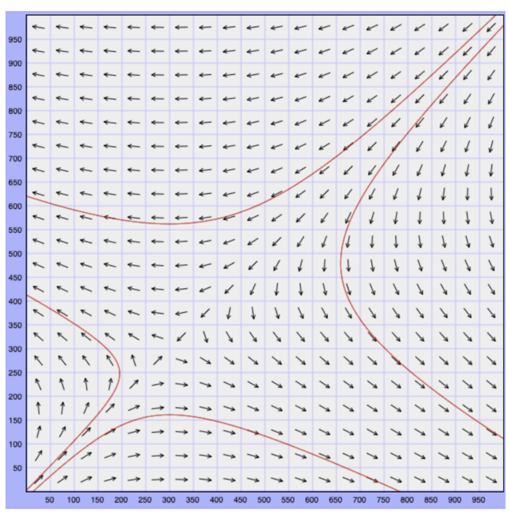

Figure 12: Direction field and trajectories for $\begin{bmatrix} m' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} m \\ w \end{bmatrix} + \begin{bmatrix} 300 \\ 300 \end{bmatrix}$

- If $\vec{g}(t) = 0$, this system is *homogeneous*

Theorem

Existence and Uniqueness: Given

$$\begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} p_{11}(t)x + p_{12}(t)y + g_1(t) \\ p_{21}(t)x + p_{22}(t)y + g_2(t) \end{bmatrix}$$

and

$$\begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

if $p_{11}, \cdots, p_{22}, g_1, \cdots, g_2$ are continous on an open interval $t_0 \in (\alpha, \beta)$, then there exists a unique solution in the interval (α, β).

Definition

A first order linear ODE

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = K(t)\vec{x} + \vec{g}(t)$$

is autonomous if coefficients do not depend on t, i.e.

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x} + \vec{b}$$

As a consequence, if a first order linear ODE is autonomous, a unique solution exists and is valid for all t.

Equilibrium Points

- The equilibrium points are where $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = 0$; to find these we need to solve the system

Definition

For a first order linear autonomous ODE

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x} + \vec{b}$$

the constant solution $\vec{x} = -A^{-1}\vec{b}$ is an equilibrium solution/critical point, assuming A^{-1} exists

- If A is not invertible, there may be either no equilibrium points or an infinite number of equilibrium points, e.g. an entire line of equilibrium points where vectors on both sides point towards the line

Lecture 9, Sep 26, 2022

Nonhomogeneous to Homogeneous

- Homogeneous ODEs always have an equilibrium at the origin, whereas nonhomogeneous ODEs' equilibrium points aren't at the origin
- The equilibrium point for $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x} + b$ is $\vec{x}_{eq} = -A^{-1}b$

- With a change of coordinates $\tilde{\vec{x}} = \vec{x} - \vec{x}_{eq}$, we get $\frac{\mathrm{d}\tilde{\vec{x}} + \vec{x}_{eq}}{\mathrm{d}t} = A(\tilde{\vec{x}} + \vec{x}_{eq}) \implies \frac{\mathrm{d}\tilde{\vec{x}}}{\mathrm{d}t} = A\tilde{\vec{x}}$, a homogeneous system of ODEs

Superposition

- We like homogeneous ODEs because we can superimpose them

Important

Principle of Superposition: Given $\vec{x}_1(t), \vec{x}_2(t)$ are solutions to $\vec{x}'(t) = A\vec{x}(t)$, then $c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$ is also a solution for any c_1, c_2

- Proof: $\frac{\mathrm{d}}{\mathrm{d}t}(c_1\vec{x}_1(t) + c_2\vec{x}_2(t)) = c_1x_1'(t) + c_2x_2'(t) = c_1Ax_1(t) + c_2Ax_2(t) = A(c_1\vec{x}_1(t) + c_2\vec{x}_2(t))$

Linear Independence of Solutions

Definition

Two solutions $\vec{x}_1(t), \vec{x}_2(t)$ are linearly dependent if $\exists k$ s.t. $\vec{x}_1(t) = k\vec{x}_2(t)$

- Given two independent solutions, we can take linear combinations of them to span the full solution space and find a solution for any initial condition
- However, if the solutions are not independent, we can't do that

Definition

The Wronskian $W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix} \vec{x}_1(t) & \vec{x}_2(t) \end{vmatrix}$

If $W[\vec{x}_1, \vec{x}_2](t) = 0$, then x_1, x_2 are linearly dependent

General Solutions Through Eigendecomposition

- Given $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$, guess $\vec{x}(t) = e^{\lambda t}\vec{v}$
  - This guess corresponds to the straight line solutions; their directions don't change, and their magnitudes change exponentially
- Substituting in, we get $\lambda\vec{v} = A\vec{v}$: if λ and $\vec{v}$ are an eigenvalue and eigenvector of A, then $\vec{x} = e^{\lambda t}\vec{v}$ solves the ODE
- Assuming $\lambda_1 \neq \lambda_2$ we have two independent solutions $\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1, \vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2$ corresponding to the two eigenvalues and eigenvectors
  - We know the Wronskian is nonzero because eigenvectors for different eigenvalues are independent
- From them we can generate the general solution $c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$, which spans the 2D space of all initial conditions

Lecture 10, Sep 29, 2022

Eigenvalues of Linear ODE Systems

- Example: $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \begin{bmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}\vec{x}, \vec{x}_0 = \begin{bmatrix} -16 \\ 20 \end{bmatrix}$

- Eigenvalues and eigenvectors: $\lambda_1 = -\frac{7}{4}, \vec{v}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}, \lambda_2 = -\frac{1}{8}, \vec{v}_2 = \frac{1}{2}$
- General solution: $\vec{x}(t) = c_1 e^{-\frac{7}{4}} \begin{bmatrix} 6 \\ -1 \end{bmatrix} + c_2 e^{-\frac{1}{8}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- Solution to IVP: $x(0) = \begin{bmatrix} -16 \\ 20 \end{bmatrix} = c_1 \begin{bmatrix} 6 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \implies c_1 = -4, c_2 = 8$
- Unstable equilibrium (sink) at $(0, 0)$; all solutions approach this as $t \to \infty$

- To visualize these in 2D, first plot the eigenvectors; use the sign of the eigenvalues to determine the directions of solutions along the eigenvectors
- For a given solution, it moves in the direction of the "dominant" eigenvector faster (the dominant eigenvector is the one with the greatest magnitude in eigenvalue)
- With two negative eigenvalues, all solutions tend towards the equilibrium at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$
  - The equilibrium at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a *sink* and stable equilibrium
- With two positive eigenvalues, all solutions (except for the one starting at the origin) diverge towards infinity
  - The equilibrium at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a *source* and unstable equilibrium
- With one positive and one negative eigenvalue, one of the eigenvectors is divergent and one is convergent
  - The equilibrium at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a semistable equilibrium
- With one negative and one zero eigenvalue, solutions converge towards the zero eigenvalue, which is an entire line of equilibrium points

Lecture 11, Sep 30, 2022

Lotka-Volterra (Predator-Prey)

- $\begin{cases} x' = \alpha x - \beta xy \\ y' = -\gamma y + \delta xy \end{cases}$
- Equilibrium point exists at $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{\delta} \\ \frac{\alpha}{\beta} \end{bmatrix}$

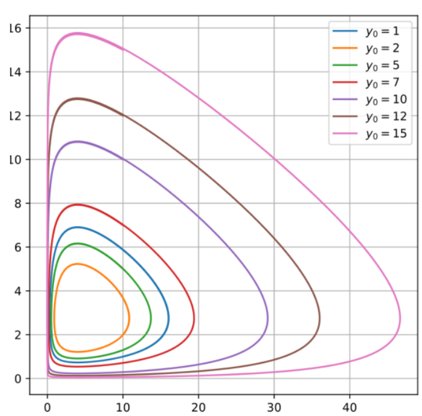

Figure 13: Solution curves to the system

- This system is nonlinear, so we have to linearize it
- We will linearize around the equilibrium

- The Jacobian evaluated is $J = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix}$
  - At equilibrium this is $\begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\gamma}{\beta} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- The eigenvalues of this system are complex! $\lambda = \pm\sqrt{\alpha\gamma}$
  - When eigenvalues are complex, solutions have spirals

Complex Eigenvalues

- Theorem: If A is a real matrix, then its eigenvalues come in complex conjugate pairs
  - Eigenvalues also come in complex conjugate pairs, e.g. if $v_1 = \begin{bmatrix} 1 - i5 \\ 2 + i \end{bmatrix}$ then $v_2 = \begin{bmatrix} 1 + i5 \\ 2 - i \end{bmatrix}$
- Suppose $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{A}$ has complex eigenvalues
  - If we follow our usual approach we would get $\boldsymbol{x}_1(t) = e^{(\mu+i\nu)t}\boldsymbol{v}_1, \boldsymbol{x}_2(t) = e^{(\mu-i\nu)t}\bar{\boldsymbol{v}}_1$, but these are not real solutions
- Let $\boldsymbol{v}_1 = \boldsymbol{a} + i\boldsymbol{b} \implies \boldsymbol{x}_1(t) = e^{\mu t}(\cos(\nu t) + i\sin(\nu t))(\boldsymbol{a} + i\boldsymbol{b})$

$$\begin{aligned} &= e^{\mu t}(\boldsymbol{a}\cos\nu t - \boldsymbol{b}\sin\nu t) + ie^{\nu t}(\boldsymbol{a}\sin\nu t + \boldsymbol{b}\cos\nu t) \\ &= \boldsymbol{u}(t) + i\boldsymbol{w}(t) \end{aligned}$$

- $\boldsymbol{u}(t)$ and $\boldsymbol{w}(t)$ form the fundamental set of solutions: $\boldsymbol{x} = c_1\boldsymbol{u}(t) + c_2\boldsymbol{w}(t)$
  - To verify this, we need to verify that they're both solutions and the Wronskian is nonzero (for now we will take this as a given)
- Example: $\boldsymbol{x}' = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} \\ 2 & -\frac{1}{2} \end{bmatrix} \boldsymbol{x} \implies \lambda_1 = \frac{3i}{2}, \vec{v}_1 = \begin{bmatrix} 5 \\ 2 - 6i \end{bmatrix}, \lambda_2 = -\frac{3i}{2}, \vec{v}_2 = \begin{bmatrix} 5 \\ 2 + 6i \end{bmatrix}$
  - $\boldsymbol{a} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$
  - $\nu = \frac{3}{2}, \mu = 0$
  - $u(t) = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \cos\left(\frac{3}{2}t\right) - \begin{bmatrix} 0 \\ -6 \end{bmatrix} \sin\left(\frac{3}{2}t\right)$
  - As $t \to \infty$ the solutions go in a cycle
  - There is a stable equilibrium at 0
  - Distinct complex eigenvalues with zero real part creates perfectly cyclical solutions

Complex Eigenvalues Cases

- Zero real part: circular solution that go nowhere as $t \to \infty$
  - Stable equilibrium at the origin
- Negative real part: solution spirals towards the origin as $t \to \infty$
  - Stable equilibrium at the origin
- Positive real part: solution spirals outwards from the origin as $t \to \infty$
  - Unstable equilibrium at the origin

Lecture 12, Oct 3, 2022

Summary of Cases of Eigenvalues

- Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - The characteristic equation is $\lambda^2 - (a + d)\lambda + ad - bc = 0 \implies \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$

  - Therefore $\lambda = \dfrac{\operatorname{tr}(A) \pm \sqrt{(\operatorname{tr}(A))^2 - 4\det(A)}}{2}$
- Let $p = \operatorname{tr} A, q = \det A$, then the sign of $p^2 - 4q$ determines the behaviour of the ODE:
  - On the parabola, $p^2 = 4q$, we get two real, equal eigenvalues
  - Below the parabola $4q < p^2$, we get two real, distinct eigenvalues
  - Above the parabola $4q > p^2$, we get two complex eigenvalues that are complements
- Recall $\det A = \lambda_1\lambda_2$, so if $\det A < 0$ the eigenvalues have different signs; if $\det A > 0$ they have the same sign
  - Below the p axis the determinant is negative so the eigenvalues have different signs, so we get saddle points (semistable equilibria)
  - Above it we get either stable or unstable equilibrium since eigenvalues have the same sign
    * Between the p axis and parabola, on the right, the trace is positive, so both eigenvalues must be positive, leading to an unstable equilibrium
    * On the left the trace is negative, so both eigenvalues must be negative, leading to a stable equilibrium

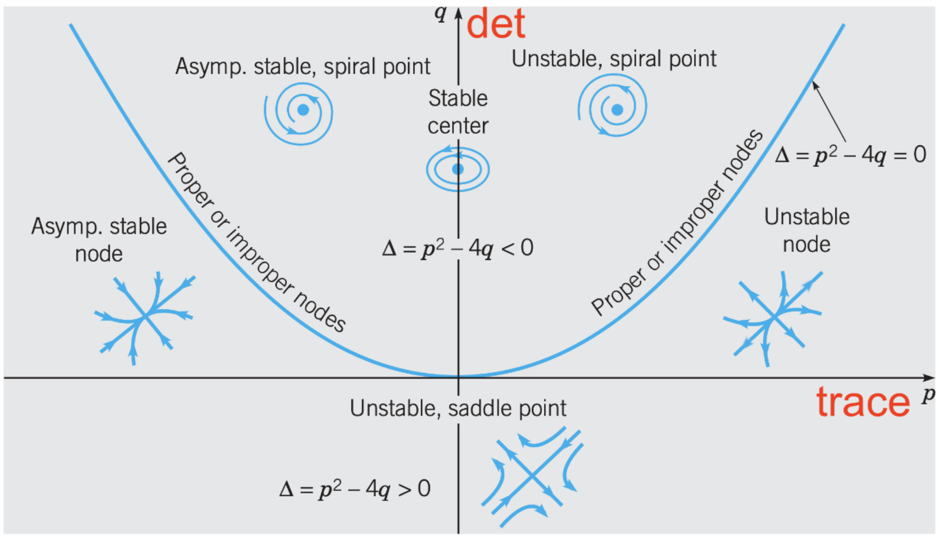

Figure 14: Summary of possible cases of the determinant and trace

Repeated Eigenvalues (and Eigenvectors)

- On the parabola we have repeated eigenvalues, e.g. $\boldsymbol{x}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{x}$
  - In this case we have repeated eigenvalues, but two distinct eigenvectors
- Another example: $\boldsymbol{x}' = \begin{bmatrix} m' \\ w' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} m \\ w \end{bmatrix}$
  - In this case we have $\lambda_1 = \lambda_2 = -\dfrac{1}{2}$, but only one eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  - Such matrices are *defective*; if we follow our usual procedure we only get one solution, which does not span the full solution space
- Notice in this system w' does not depend on m, so we can solve it independently to get $w = c_2 e^{-\frac{t}{2}}$
  - Substituting this back in we get $m' = -\dfrac{1}{2}m + c_2 e^{-\frac{t}{2}}$ which is a FO linear ODE
  - Using integrating factors we get $m = c_2 t e^{-\frac{t}{2}} + c_1 e^{-\frac{t}{2}}$
  - The final solution is $\boldsymbol{x} = c_1 e^{-\frac{t}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left(t e^{-\frac{t}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-\frac{t}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

* But wait, where did $\boldsymbol{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ come from? What does it mean?

- Our solution has the form $\boldsymbol{x}_2 = te^{-\frac{t}{2}}\boldsymbol{v} + te^{-\frac{t}{2}}\boldsymbol{w}$
  - Substituting this in: $e^{-\frac{t}{2}}\boldsymbol{v} - \frac{1}{2}te^{-\frac{t}{2}}\boldsymbol{v} - \frac{1}{2}e^{-\frac{t}{2}}\boldsymbol{w} = \boldsymbol{A}\left(te^{-\frac{t}{2}}\boldsymbol{v} + te^{-\frac{t}{2}}\boldsymbol{w}\right)$
  - Notice for this to hold we must have $-\frac{1}{2}te^{-\frac{t}{2}}\boldsymbol{v} = \boldsymbol{A}te^{-\frac{t}{2}}$ and $e^{-\frac{t}{2}}\boldsymbol{v} - \frac{1}{2}e^{-\frac{t}{2}}\boldsymbol{w} = \boldsymbol{A}e^{-\frac{t}{2}}\boldsymbol{w}$
  - This gives us $\left(\boldsymbol{A} + \frac{1}{2}\boldsymbol{I}\right)\boldsymbol{v} = 0$ and $\left(\boldsymbol{A} + \frac{1}{2}\boldsymbol{I}\right)\boldsymbol{w} = \boldsymbol{v}$
    * $\left(\boldsymbol{A} + \frac{1}{2}\boldsymbol{I}\right)\boldsymbol{w} = \boldsymbol{v}$ is a *generalized eigenvector equation*, where $\boldsymbol{w}$ is the *generalized eigenvector*
    * Solving this gives us $\boldsymbol{w} = \begin{bmatrix} k \\ 1 \end{bmatrix}$, so we can choose $k = 0$ and form our solution

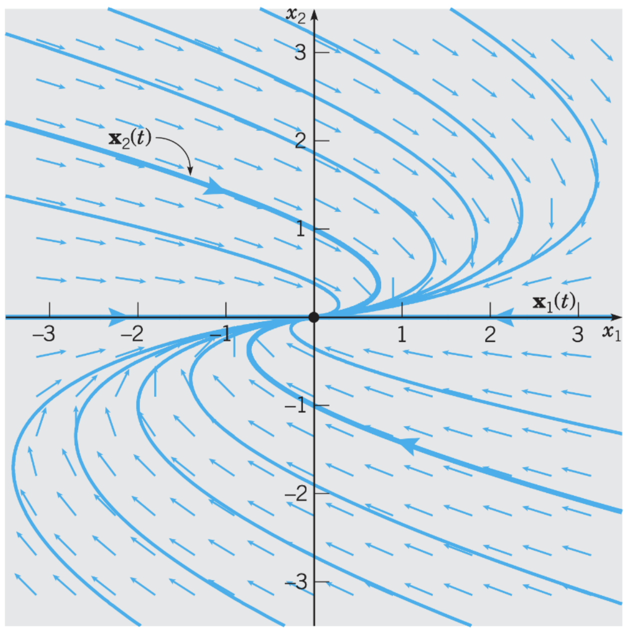

Figure 15: Solution to the system $\boldsymbol{x}' = \begin{bmatrix} m' \\ w' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} m \\ w \end{bmatrix}$

Definition

The generalized eigenvector is a vector that satisfies $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{w} = \boldsymbol{v}$, where $\boldsymbol{v}$ is the repeated eigenvector and λ is the repeated eigenvalue

Summary

When eigenvalues and eigenvectors are equal:

1. Write the first solution $\boldsymbol{x}_1(t) = e^{\lambda t}\boldsymbol{v}$ where $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v} = 0$
2. Write the second solution $\boldsymbol{x}_2(t) = te^{\lambda t}\boldsymbol{v} + e^{\lambda t}\boldsymbol{w}$ where $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{w} = \boldsymbol{v}$
3. The general solution is then $\boldsymbol{x} = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t)$

This works even when A is not triangular

Lecture 13, Oct 6, 2022

Repeated Eigenvalues Examples

- Example: $\boldsymbol{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \boldsymbol{x}, \lambda_1 = \lambda_2 = 2, \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
  - Generalized eigenvector: $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{w} = \boldsymbol{v} \implies \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \boldsymbol{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \implies \boldsymbol{w} = \begin{bmatrix} k \\ -1-k \end{bmatrix}$
  - General solution: $\boldsymbol{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \left(te^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t} \begin{bmatrix} k \\ -1-k \end{bmatrix} \right)$
    * Notice that the term with $\boldsymbol{w}$ is $ke^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$; the former can be absorbed into the c_1 term
    * We can also just choose k to be whatever we want
  - Simplified, $\boldsymbol{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \left(te^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$
  - As $t \to \infty$ the solution is dominated by the te^{2t} term or the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
  - As $t \to -\infty$ the solution goes to 0, but is still dominated by te^{2t}
  - The equilibrium at 0 is unstable; it is an *improper* equilibrium (also known as an improper node as all solutions emerge from it)

Lecture 14, Oct 7, 2022

Euler's Method

- Iterative method: Solve $y' = f(t, y)$ by $y_{n+1} = y_n + hf(t_n, y_n)$

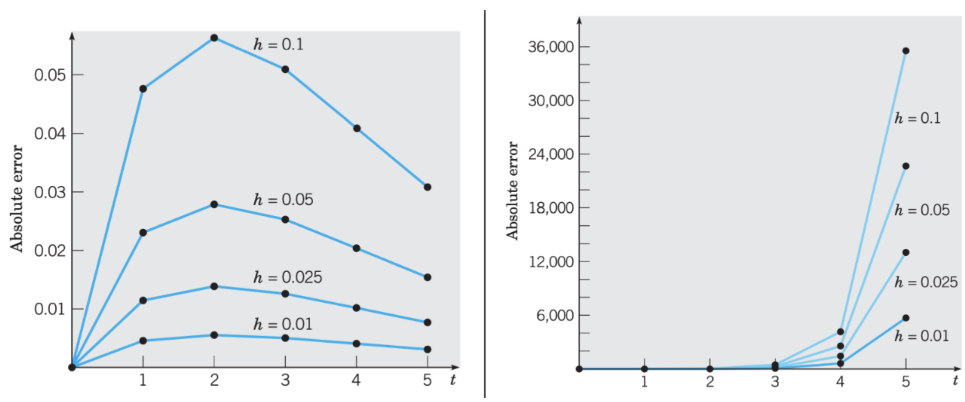

Figure 16: Comparison of errors in Euler's method for $y' = \frac{3}{2} - t - \frac{1}{2}$ and $y' = 4 - t + 2y$

- A larger step size always increases the error in Euler's method
- Error accumulates in Euler's method, but this is not always the case – sometimes the absolute error can decrease
  - Why do approximations work better sometimes?

- Notice the left has a solution of $y(t) = 7 - 2t - Ce^{-\frac{t}{2}}$ and the right has a solution of $y(t) = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}$
  - In the left one, all solutions converge towards $y(t) = 7 - 2t$, while in the right solution diverge
  - A small error in the left eventually decays away, while a small error in the right blows up

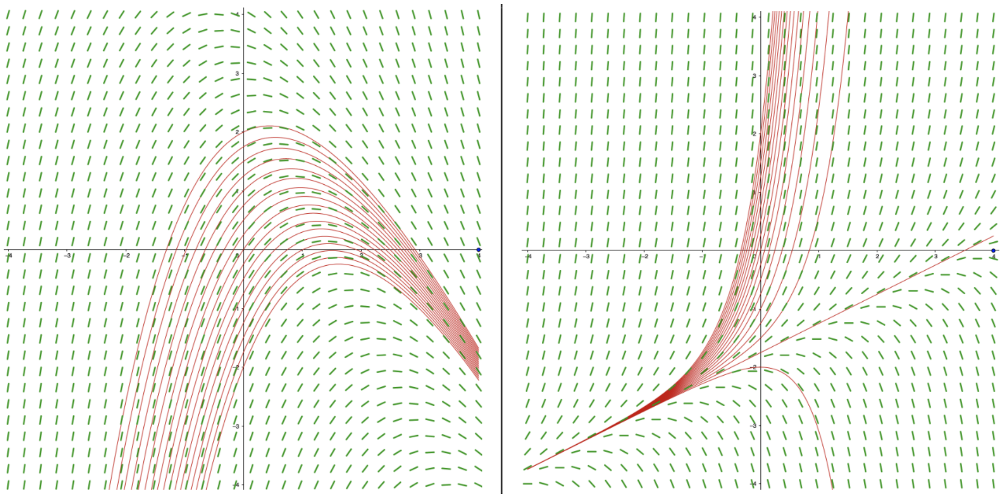

Figure 17: Comparison of solutions for $y' = \frac{3}{2} - t - \frac{1}{2}$ and $y' = 4 - t + 2y$

Lecture 15, Oct 13, 2022

Errors in Numerical Approximations

- Round-off errors
- Euler's method relies on successive linear approximations
- Global truncation error: $E_n = \phi(t_n) - y_n$, error accumulated across all steps
  - We use y_n instead of $\phi(t_n)$ to determine y_{n+1} so errors can accumulate
- Local truncation error: Error due to the linear approximation only

Local Truncation Error

- Consider a general ODE $y' = f(t, y)$ with solution $\phi(t)$, so $\phi'(t) = f(t, \phi(t))$
- With Euler's method, $y_{n+1} = y_n + hf(t_n, y_n)$
- The error is $|y_{n+1} - \phi(t_{n+1})|$
- Using a Taylor approximation: $\phi(t_{n+1}) = \phi(t_n) + \phi'(t_n)h + \frac{1}{2}\phi''(\bar{t}_n)h^2$ where $t_n < \bar{t}_n < t_n + h$
  - This is an exact equality due to $\bar{t}_n$ (Taylor's Remainder Theorem)
  - Note we assumed that ϕ is twice-differentiable and continuous in its derivatives
- The error is $|\phi(t_{n+1} - y_{n+1})| = (\phi(t_n) - y_n) + h(f(t_n, \phi(t_n)) - f(t_n, y_n)) + \frac{1}{2}\phi''(\bar{t}_n)h^2$
  - The term $\phi(t_n) - y_n = 0$ because this is a local error
  - Since $\phi(t_n) = y_n$ the middle term is also 0
  - Therefore the error is $\frac{1}{2}\phi''(\bar{t}_n)h^2$
- To bound $\frac{1}{2}\phi''(\bar{t}_n)h^2$, we assume $|\phi''(t)| \leq M$ so $|e_n| \leq \frac{Mh^2}{2}$

Global Truncation Error

- The number of steps is $n = \frac{T - t_0}{h}$
- The global truncation error can be approximated as $n\frac{Mh^2}{2} = \frac{(t - t_0)Mh}{2}$
- Notice the global truncation error decreases linearly with h
  - Euler's method is a first-order method because the power of h is 1

Assumptions

- For ϕ'' to be continuous to invoke the Taylor series, we need $\frac{\partial f}{\partial t}(t,\phi(t))+\frac{\partial f}{\partial y}(t,\phi(t))f(t,\phi(t))$ continuous since $\phi' = f(t,\phi(t))$
  - This means we need f, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous

Lecture 16, Oct 14, 2022

Numerical Integration Methods

- Riemann sums approximates the function as a series of constant value segments
- Trapezoidal rule approximates the function as a number of linear segments
- Simpson's one-third rule approximates the function as a series of parabolas
  - Take the current point, the next point, a point halfway, and fit a parabola
  - $\int_a^b f(x)\,\mathrm{d}x \approx \int a^b P(x)\,\mathrm{d}x = \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$
    * This is just a closed-form solution for the integral of the parabola that passes through these 3 points

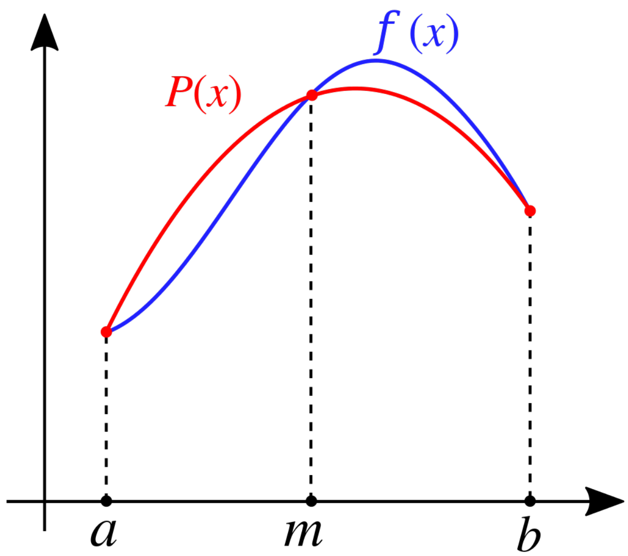

Figure 18: Simpson's Rule

Improved Euler Method

- When we solve an ODE, we are essentially integrating: $\phi(t_{n+1}) = \phi(t_n) + \int_{t_n}^{t_{n+1}} f(t,\phi(t))\,\mathrm{d}t$
- Euler's method, $y_{n+1} = y_n + hf(t_n, y_n)$ is essentially approximating f as a constant value $f(t,y) = f(t_n, y_n)$
  - This essentially makes a Riemann sum – so what if we used a trapezoidal sum instead?
- This leads to the improved Euler method $y_{n+1} = y_n + h\frac{f(t_n,y_n)+f(t_{n+1},y_{n+1})}{2}$
  - However, we can't quite use y_{n+1} in the right hand side because we haven't found it yet
  - We can use Euler's method to find an estimate for it

Definition

The Improved Euler/Heun/~~Abdullah~~ Method:

$$y_{n+1} = y_n + h\frac{f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))}{2}$$

- IEM is a *second order* method – local truncation error is $O(h^3)$ and global truncation error is $O(h^2)$
  - However, IEM requires two function evaluations per step
  - But if $h \ll \frac{1}{2}$ this still makes IEM much more efficient

Runge-Kutta Method

Definition

The Runge-Kutta Method:

$$y_{n+1} = y_n + h\frac{s_{n1} + 2s_{n2} + 2s_{n3} + s_{n4}}{6}$$

where

$$\begin{aligned} s_{n1} &= f\left(t_n, y_n\right) \\ s_{n2} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hs_{n1}\right) \\ s_{n3} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hs_{n2}\right) \\ s_{n4} &= f\left(t_n + h, y_n + hs_{n3}\right) \end{aligned}$$

- The Runge-Kutta method essentially approximates with a parabola (Simpson's Rule)
- The derivative is evaluated at the current point, the next point, and also the point in the middle
- Runge-Kutta is a *fourth order* method – local truncation error is $O(h^5)$ and global truncation error is $O(h^4)$
  - Each step requires 4 function evaluation

Adaptive Step Sizes

- What if we could use smaller step sizes where it's needed?
- Run a standard step of Euler's method, and one IEM step; if we assume that the IEM gives the absolute truth, then the difference between the two approximations is the error
- The local truncation error should scale like h^2
  - $\frac{e_{n+1}^{\text{est}}}{h^2} = \frac{|y_{n+1}^{\text{euler}} - y_{n+1}^{\text{IEM}}|}{h^2} \approx \text{const}$
- If we adjust the step size to h_{new}, with some new local truncation error ϵ, then $\frac{\epsilon}{h_{\text{new}}^2} \approx \text{const}$
- Therefore if we want to keep the local truncation error ϵ constant, then we can have $\frac{e_{n+1}^{\text{est}}}{h^2} = \frac{\epsilon}{h_{\text{new}}^2}$

Important

To keep the error rougly fixed at ϵ, adjust the step size as

$$h_{\text{new}} = h\sqrt{\frac{\epsilon}{e^{\text{est}}_{n+1}}}$$

where $e^{\text{est}}_{n+1} = |y^{\text{euler}}_{n+1} - y^{\text{IEM}}_{n+1}|$

Lecture 17, Oct 17, 2022

First Order Linear Systems of Dimension n

- A general first order linear system can be described by $\vec{\boldsymbol{x}}' = \boldsymbol{P}(t) + \vec{\boldsymbol{g}}(t)$

Theorem

Given

$$\vec{\boldsymbol{x}}' = \boldsymbol{P}(t) + \vec{\boldsymbol{g}}(t), \vec{\boldsymbol{x}}(t_0) = \vec{\boldsymbol{y}_0}$$

and $\boldsymbol{P}(t), \boldsymbol{g}\vec{(t)}$ continuous over $t_0 \in (\alpha, \beta)$, then there exists a unique solution in the interval (α, β)

- Note $\boldsymbol{P} : \mathbb{R} \mapsto {}^n\mathbb{R}^n$ and $\vec{\boldsymbol{g}} : \mathbb{R} \mapsto \mathbb{R}^n$
- We can always centre the problem, so from now we assume $\vec{\boldsymbol{g}}(t) = 0$

Theorem

Principle of Superposition: Given

$$\vec{\boldsymbol{x}}' = \boldsymbol{P}(t)\vec{\boldsymbol{x}}$$

and $\vec{\boldsymbol{x}}_1(t), \vec{\boldsymbol{x}}_2(t), \cdots, \vec{\boldsymbol{x}}_n(t)$ are solutions, then

$$c_1\vec{\boldsymbol{x}}_1(t) + c_2\vec{\boldsymbol{x}}_2(t) + \cdots + c_n\vec{\boldsymbol{x}}_n(t)$$

is also a solution for any $c_1, c_2, \cdots, c_n$

This makes the set of all solutions a vector space

Definition

Functions $\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \cdots, \vec{\boldsymbol{x}}_n$ are linearly independent if

$$c_1\vec{\boldsymbol{x}}_1(t) + c_2\vec{\boldsymbol{x}}_2(t) + \cdots + c_n\vec{\boldsymbol{x}}_n(t) = \vec{0} \implies c_1 = c_2 = \cdots = c_n = 0$$

Definition

The Wronskian

$$W[\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \cdots, \vec{\boldsymbol{x}}_n](t) = \det \begin{bmatrix} | & | & & | \\ \vec{\boldsymbol{x}}_1 & \vec{\boldsymbol{x}}_2 & \cdots & \vec{\boldsymbol{x}}_n \\ | & | & & | \end{bmatrix} = \det(\boldsymbol{X}(t))$$

If $\det(\boldsymbol{X}(t)) \neq 0$ for all t, then the solutions are independent

Theorem

If $\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \cdots, \vec{\boldsymbol{x}}_n$ are linearly independent solutions, then any solution can be expressed as

$$\vec{\boldsymbol{x}}(t) = c_1\vec{\boldsymbol{x}}_1(t) + c_2\vec{\boldsymbol{x}}_2(t) + \cdots + c_n\vec{\boldsymbol{x}}_n(t)$$

for some set of unique $c_1, c_2, \cdots, c_n$

Such a set of $\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \cdots, \vec{\boldsymbol{x}}_n$ is known as the *fundamental set* of the ODE

- This set of $c_1, c_2, \cdots, c_n$ depends on the initial conditions: If $\vec{\boldsymbol{x}}(0) = \vec{\boldsymbol{b}}$, then $\vec{\boldsymbol{c}} = \boldsymbol{X}(0)^{-1}\vec{\boldsymbol{b}}$

Lecture 18, Oct 20, 2022

The Matrix Exponential

- If a scalar IVP $x' = ax, x(0) = x_0$ can be solved by $x = e^{at}x_0$, then can we solve $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0$ with $\boldsymbol{x} = e^{\boldsymbol{A}t}\boldsymbol{x}_0$?
- We can define the matrix exponential $e^{\boldsymbol{A}t}$ using a Taylor series, similar to a scalar exponential

Definition

The matrix exponential

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2t^2 + \frac{1}{3!}\boldsymbol{A}^3t^3 + \cdots = \sum_{k=0}^{\infty}\frac{\boldsymbol{A}^kt^k}{k!}$$

- The matrix exponential has the same properties as the scalar exponential
  - $e^{\boldsymbol{0}t} = \boldsymbol{I}$
  - $\frac{\mathrm{d}}{\mathrm{d}t}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t}$
  - $e^{\boldsymbol{A}(t+\tau)} = e^{\boldsymbol{A}t}e^{\boldsymbol{A}\tau}$
  - $(e^{\boldsymbol{A}t})^{-1} = e^{-\boldsymbol{A}t}$
  - $Ae^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}$
  - $e^{(\boldsymbol{A}+\boldsymbol{B})t} = e^{\boldsymbol{A}t}e^{\boldsymbol{B}t}$, but only if $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$

Theorem

Given an ODE $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$,

$$\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t}$$

is a solution to this ODE, and satisfies $\boldsymbol{\Phi}(0) = \boldsymbol{I}$

- Note that this is a matrix differential equation; this contains multiple solutions $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots$, which forms the fundamental set
- The matrix exponential is also sometimes known as the *special fundamental matrix*, because its columns are solutions that form a basis for the solution space
  - The general solution $\boldsymbol{x}(t)$ can be written as $\boldsymbol{x}(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) + \cdots = \boldsymbol{\Phi}(t)\boldsymbol{c}$
- If we are given an initial condition $\boldsymbol{X}(0) = \boldsymbol{x}_0$, then $\boldsymbol{x}(0) = \boldsymbol{\Phi}(0)\boldsymbol{c} \implies \boldsymbol{c} = \boldsymbol{\Phi}(0)^{-1}$; since $\boldsymbol{\Phi}(0) = \boldsymbol{I}$, the IVP is satisfied by $\boldsymbol{X}(t) = \boldsymbol{x}_0\boldsymbol{\Phi}(t)$

Theorem

The IVP

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0$$

is satisfied by

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0$$

Calculating The Matrix Exponential

- How do we take all those higher powers of $\boldsymbol{A}$?
- Eigendecomposition: $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}$ where $\boldsymbol{D}$ is a diagonal matrix of the eigenvalues and $\boldsymbol{V}$ is a matrix of all the eigenvectors
  - This works because $\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{D} = \boldsymbol{D}\boldsymbol{V}$
- Using eigendecomposition we can easily take higher powers: $\boldsymbol{A}^k = \boldsymbol{V}\boldsymbol{D}^k\boldsymbol{V}^{-1}$
  - This is great because $\boldsymbol{D}$ is diagonal, so $\boldsymbol{D}^k$ simply has the diagonal entries to the power of k
  - This allows us to write $e^{\boldsymbol{A}t} = \boldsymbol{V}e^{\boldsymbol{D}t}\boldsymbol{V}^{-1}$, and $e^{\boldsymbol{D}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots \\ 0 & e^{\lambda_2 t} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$

Important

The IVP

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0$$

is satisfied by

$$\boldsymbol{x} = e^{\boldsymbol{A}t}\boldsymbol{x}_0 = \boldsymbol{V}e^{\boldsymbol{D}t}\boldsymbol{V}^{-1}\boldsymbol{x}_0$$

Lecture 19, Oct 21, 2022

Second Order Linear Differential Equations

- A second order linear ODE can be expressed as $y'' = f(t, y, y')$
  - Initial conditions $y(t_0) = y_0, y'(t_0) = y_1$
  - Notice two initial conditions are needed because we have 2 integration constants
- A second order linear ODE can be expressed as $y'' + p(t)y' + q(t)y = g(t)$
- A second order ODE can be written in terms of two first order ODEs:
  - Define $x_1 = y, x_2 = y'$
  - $y'' = x_2' = f(t, x_1, x_2)$
  - $y' = x_2 = x_1' \implies x_1' = x_2$
- If we had a second order linear ODE $y'' + p(t)y' + q(t)y = g(t)$, we can write it as a system of linear ODEs:
  - $x_1' = x_2, x_2' = -q(t)x_1 - p(x)x_2 + g(t)$
  - $\vec{\boldsymbol{x}}' = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}\vec{\boldsymbol{x}} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$
  - Initial conditions can be added as $\vec{\boldsymbol{x}}(t_0) = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$

Theorem

Existence and Uniqueness Theorem: Given

$$y'' + p(t)y' + q(t)y = g(t)$$

if $p(t), q(t), g(t)$ are continuous over $t_0 \in (\alpha, \beta)$, then there exists a unique solution over (α, β)

- Second order linear ODEs also obey the theorem of superposition

Lecture 20, Oct 24, 2022

Second Order Linear Homogeneous Autonomous ODE

- Consider the ODE: $ay'' + by' + cy = 0$
- Recall we can express this as $\boldsymbol{x} = \begin{bmatrix} y \\ y' \end{bmatrix}, \boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \boldsymbol{x}$
  - The eigenvalues are: $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0 \implies a\lambda^2 + b\lambda + c = 0 \implies \lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The eigenvectors are $\boldsymbol{v} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$
- 3 cases for these eigenvalues:
  - Real and distinct (overdamped)
    * General solution $\boldsymbol{x} = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$, which gives $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ when λ are distinct
  - Real and equal (critically damped)
    * $\lambda_1 = \lambda_2 = -\dfrac{b}{2a}$
    * We only have a single eigenvector $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$, so we need to find the generalized eigenvector
    * $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{w} = \boldsymbol{v} \implies \boldsymbol{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
    * This gives us $\boldsymbol{x} = c_1 e^{\lambda_1 t} \boldsymbol{v}_1 + c_2 e^{\lambda_1 t}(t\boldsymbol{v}_1 + \boldsymbol{w}_1) = c_1 e^{\lambda t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda t} \begin{bmatrix} t \\ \lambda t + 1 \end{bmatrix}$
    * This gives $y = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$
  - Complex conjugates (underdamped)
    * $\lambda_1 = \mu + i\nu, \lambda_2 = \mu - i\nu$ where $\mu = -\dfrac{b}{2a}, \nu = \dfrac{\sqrt{4ac - b^2}}{2a}$
    * Construct the solution just like before and use Euler's identity: $\boldsymbol{x} = c_1 e^{(\mu + i\nu)t} \begin{bmatrix} 1 \\ \mu + i\nu \end{bmatrix} + c_2 e^{(\mu - i\nu)t} \begin{bmatrix} 1 \\ \mu - i\nu \end{bmatrix} = c_1 e^{\mu t} \begin{bmatrix} \cos \nu t \\ \mu \cos \nu t - \nu \sin \nu t \end{bmatrix} + c_2 e^{\mu t} \begin{bmatrix} \sin \nu t \\ \mu \sin \nu t + \nu \cos \nu t \end{bmatrix}$
    * This gives: $y = c_1 e^{\mu t} \cos \nu t + c_2 e^{\mu t} \sin \nu t$
- Example: $y'' + 5y' + 6y = 0$
  - Eigenvalues are $\lambda_1 = -2, \lambda_2 = -3$, real and distinct
  - Eigenvectors are then $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$
- Example: $y'' + y' + y = 0$
  - Eigenvalues are complex conjugates: $\lambda = -\dfrac{1}{2} \pm i\sqrt{\dfrac{3}{2}}$
- Example: $y'' - y' + \dfrac{1}{4}y = 0$

– Eigenvalues are equal: $\lambda_1 = \lambda_2 = \dfrac{1}{2}$

Summary

The ODE $ay'' + by' + cy = 0$ is solved by:

1. When λ are real and distinct:
$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
2. When λ are complex conjugates:
$$y = c_1 e^{\mu t} \cos \nu t + c_2 e^{\mu t} \sin \nu t$$
3. When λs are real and equal:
$$y = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$$

where
$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
and $\lambda = \mu \pm i\nu$ when λ are complex

Lecture 21, Oct 27, 2022

Second Order Linear Nonhomogeneous ODEs

- Recall: All solutions to $\boldsymbol{Ax} = \boldsymbol{b}$ can be constructed by taking a particular solution to it and adding a solution to $\boldsymbol{Ax} = \boldsymbol{0}$
- Consider a nonhomogeneous second order linear ODE $ay'' + by' + cy = g(t)$ and its homogeneous counterpart $ay'' + by + cy = 0$
- 2 observations:
  1. Let y_h be a solution to the homogeneous ODE and y_p be a particular solution to the nonhomogeneous ODE, then $y_h + y_p$ solves the nonhomogeneous ODE
  2. Let $y_p, \hat{y}_p$ be two particular solutions to the nonhomogeneous ODE, then $\hat{y}_p - y_p = y_h$ solves the homogeneous ODE
- This means that given a nonhomogeneous ODE, we simply have to find a particular solution to it y_p, and the general solution y_h to the homogeneous ODE; then the general solution to the nonhomogeneous ODE is the sum of the two

Important

Given
$$ay'' + by' + cy = g(t)$$
the general solution can be found by
$$y_g = y_h + y_p$$
where y_h is the general solution to the homogeneous ODE $ay'' + by + cy = 0$ and y_p is one particular solution to the homogeneous ODE

Method of Undetermined Coefficients

- Involves guessing "trial solutions" based on the form of $g(t)$
  - For e^{rt} we guess Ae^{rt}
  - For $\sin(\omega t)$ or $\cos(\omega t)$ we guess $A\sin(\omega t) + B\cos(\omega t)$
  - For degree n polynomial we guess $B_0 + B_1 t + B_2 t^2 + \cdots + B_n t^n$

  - For a combination of these, we guess a combination of the corresponding guesses
  - If the guess solution appears in the homogeneous solution, multiply by t
- Example: $y'' - 3y - 4y = 3e^{2t}$
  - $\lambda = -1, 4$
  - General homogeneous solution: $y_h(t) = c_1 e^{-t} + c_2 e^{4t}$
  - Guess: $y = Ae^{2t}$
    * Plug this in we get $4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$
    * $-6A = 3 \implies A = -\frac{1}{2}$
  - Particular solution: $y_p = -\frac{1}{2}e^{2t}$
  - General solution: $y = -\frac{1}{2}e^{2t} + c_1 e^{-t} + c_2 e^{4t}$
- Example: $y'' - 3y' - 4y = 2e^{-t}$
  - General homogeneous solution: $y_h(t) = c_1 e^{-t} + c_2 e^{4t}$
  - Guess: $y = Ae^{-t}$
    * Plugging this in: $Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} = 2e^{-t}$
    * However this is equal to zero! This is because e^{-t} is already in our homogeneous solution
  - Guess: $y = Ate^{-t}$
    * $y' = Ae^{-t} - Ate^{-t}$
    * $y'' = -2Ae^{-t} + Ate^{-t}$
    * Plugging this in and solving we get $A = -\frac{2}{5}$

Example Problem: RLC Circuit

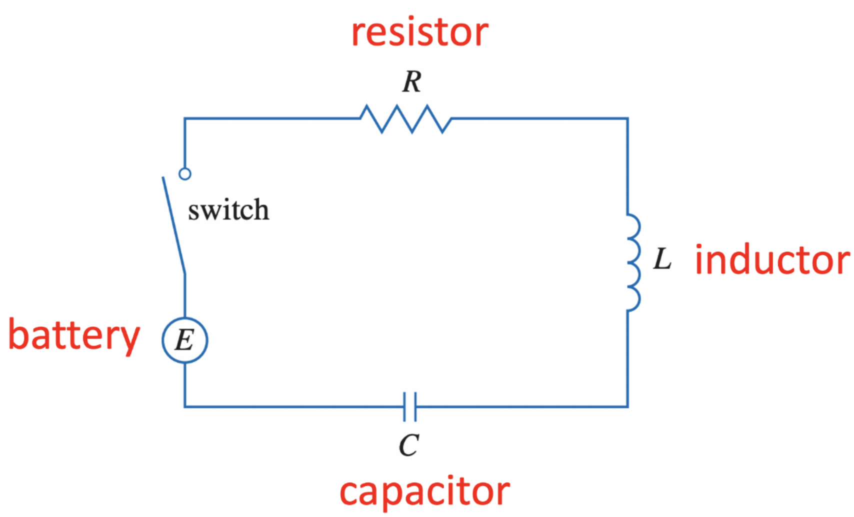

Figure 19: RLC Circuit

- Voltage across the inductor is $L\frac{\mathrm{d}I}{\mathrm{d}t}$; voltage across a capacitor is $\frac{Q}{C}$
- From Kirchhoff's Voltage Law: $L\frac{\mathrm{d}I}{\mathrm{d}t} + RI + \frac{Q}{C} = E(t)$
  - Since $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$ we can transform this into $L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = E(t)$
  - This is a second order linear nonhomogeneous ODE

Lecture 22, Oct 28, 2022

Method of Variation of Parameters

- Consider a nonhomogeneous system: $\vec{\boldsymbol{x}}' = \boldsymbol{P}(t)\vec{\boldsymbol{x}} + \vec{\boldsymbol{g}}(t)$
  - We solve the homogeneous system $\vec{\boldsymbol{x}}' = \boldsymbol{P}(t)\vec{\boldsymbol{x}}$
  - To do this we need a fundamental set $\{\, \vec{\boldsymbol{x}}_1(t), \vec{\boldsymbol{x}}_2(t) \,\}$
    - This means $\vec{\boldsymbol{x}}_1' = \boldsymbol{P}(t)\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2' = \boldsymbol{P}(t)\vec{\boldsymbol{x}}_2$
  - We can write $\boldsymbol{X}' = \boldsymbol{P}(t)\boldsymbol{X}$, where $\boldsymbol{X} = \begin{bmatrix}\vec{\boldsymbol{x}}_1 & \vec{\boldsymbol{x}}_2\end{bmatrix}$
  - From this we can construct a general solution of the homogeneous system: $\vec{\boldsymbol{x}} = c_1\vec{\boldsymbol{x}}_1(t) + c_2\vec{\boldsymbol{x}}_2(t)$
- Guess solution: $u_1(t)\vec{\boldsymbol{x}}_1(t) + u_2(t)\vec{\boldsymbol{x}}_2(t) = \boldsymbol{X}(t)\vec{\boldsymbol{u}}(t)$ where $\vec{\boldsymbol{u}} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$
- Substitute into nonhomogeneous equation: $(\boldsymbol{X}(t)\vec{\boldsymbol{u}}(t))' = \boldsymbol{X}'(t)\vec{\boldsymbol{u}}(t) + \boldsymbol{X}(t)\vec{\boldsymbol{u}}'(t) = \boldsymbol{P}(t)\boldsymbol{X}(t)\vec{\boldsymbol{u}}(t) + \vec{\boldsymbol{g}}(t)$
  - Since $\boldsymbol{X}'(t) = \boldsymbol{P}(t)\boldsymbol{X}(t)$ this simplifies to just $\boldsymbol{X}(t)\vec{\boldsymbol{u}}'(t) = \vec{\boldsymbol{g}}(t)$
- Therefore $\vec{\boldsymbol{u}}'(t) = \boldsymbol{X}(t)^{-1}\vec{\boldsymbol{g}}(t)$
  - We know $\boldsymbol{X}(t)$ is invertible since the fundamental matrix always has a nonzero Wronskian
- Now we can integrate: $\vec{\boldsymbol{u}}(t) = \vec{\boldsymbol{c}} + \displaystyle\int \boldsymbol{X}(t)^{-1}\vec{\boldsymbol{g}}(t)\,\mathrm{d}t$
- $\vec{\boldsymbol{x}} = \boldsymbol{X}(t)\vec{\boldsymbol{u}}(t) = \boldsymbol{X}(t)\vec{\boldsymbol{c}} + \boldsymbol{X}(t)\displaystyle\int \boldsymbol{X}(t)^{-1}\vec{\boldsymbol{g}}(t)\,\mathrm{d}t$
  - Notice this consists of $\boldsymbol{X}(t)\vec{\boldsymbol{c}}$, which is the general solution to the homogeneous equation, plus a particular solution to the nonhomogeneous equation
- Note: For a 2x2 problem, $\boldsymbol{X}(t)^{-1} = \dfrac{1}{W[\vec{\boldsymbol{x}}_1(t), \vec{\boldsymbol{x}}_2(t)]}\begin{bmatrix} x_{22}(t) & -x_{12}(t) \\ -x_{21}(t) & x_{11}(t) \end{bmatrix}$

Theorem

The general linear nonhomogeneous system

$$\vec{\boldsymbol{x}}' = \boldsymbol{P}(t)\vec{\boldsymbol{x}} + \vec{\boldsymbol{g}}(t)$$

is solved by

$$\vec{\boldsymbol{x}} = \boldsymbol{X}(t)\vec{\boldsymbol{u}}(t) = \boldsymbol{X}(t)\vec{\boldsymbol{c}} + \boldsymbol{X}(t)\int \boldsymbol{X}(t)^{-1}\vec{\boldsymbol{g}}(t)\,\mathrm{d}t$$

where $\boldsymbol{X}(t)$ is the fundamental matrix of the system, and $\vec{\boldsymbol{c}}$ is a vector of constants determined by initial conditions

Second Order Nonhomogeneous ODE

- Consider the ODE $y'' + p(t)y' + q(t)y = g(t)$
  - We can use variation of parameters to solve this
- Convert to system: $\vec{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}\vec{\boldsymbol{x}} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$
- Notice the fundamental matrix has the structure $\boldsymbol{X}(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$
- The particular solution: $\boldsymbol{X}(t)\displaystyle\int \boldsymbol{X}(t)^{-1}\vec{\boldsymbol{g}}(t)\,\mathrm{d}t = \dfrac{1}{W[\vec{\boldsymbol{y}}_1(t), \vec{\boldsymbol{y}}_2(t)]}\begin{bmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{bmatrix}\begin{bmatrix} 0 \\ g(t) \end{bmatrix}\,\mathrm{d}t$

$$= \boldsymbol{X}(t)\int \frac{1}{W[\vec{\boldsymbol{y}}_1(t), \vec{\boldsymbol{y}}_2(t)]}\begin{bmatrix} -g(t)y_2(t) \\ g(t)y_1(t) \end{bmatrix}\,\mathrm{d}t$$

$$= -\begin{bmatrix} y_1(t) \\ y_1'(t) \end{bmatrix}\int \frac{1}{W}g(t)y_2(t)\,\mathrm{d}t + \begin{bmatrix} y_2(t) \\ y_2'(t) \end{bmatrix}\int \frac{1}{W}g(t)y_1(t)\,\mathrm{d}t$$

- We can now extract a particular solution for y: $y_p = -y_1(t)\displaystyle\int \frac{1}{W}g(t)y_2(t)\,\mathrm{d}t + y_2(t)\int \frac{1}{W}g(t)y_1(t)\,\mathrm{d}t$

Theorem

A particular solution to the general second order linear nonhomogeneous ODE

$$y'' + p(t)y' + q(t)y = g(t)$$

is

$$y_p = -y_1(t)\int \frac{1}{W} g(t)y_2(t)\,\mathrm{d}t + y_2(t)\int \frac{1}{W} g(t)y_1(t)\,\mathrm{d}t$$

where $\{ y_1, y_2 \}$ is the fundamental set of the homogeneous solution and

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Example

- $y'' + 4y = \dfrac{3}{\sin t}$
- Homogeneous solution:
  - $\lambda = \pm 2i$
  - Use the formula for the complex case: $y = c_1 e^{\mu t}\cos(\nu t) + c_2 e^{\mu t}\sin(\nu t)$
  - $y_1 = \cos(2t), y_2 = \sin(2t)$
- By variation of parameters, $y = c_1\cos(2t) + c_2\sin(2t) - \dfrac{\cos(2t)}{2}\displaystyle\int \frac{3}{\sin t}\sin(2t)\,\mathrm{d}t + \dfrac{\sin(2t)}{2}\displaystyle\int \frac{3}{\sin t}\cos(2t)\,\mathrm{d}t$

Lecture 23, Oct 31, 2022

The Laplace Transform

Definition

The Laplace transform of a function $f(t)$ is

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

- The Laplace transform is analogous to a change of coordinates in linear algebra
  - We're taking a function $f(t)$ to get back another function $F(s)$
  - This integral of the product of functions is akin to a dot product, but for functions; e^{-st} is a basis
    - We like a basis of e^{-st} because its derivative is proportional to itself

Example Transforms

- $$\begin{aligned}\mathcal{L}\{1\} &= \int_0^\infty e^{-st}\,\mathrm{d}t \\ &= \left[-\frac{1}{s}e^{-st}\right]_0^\infty \\ &= \lim_{A\to\infty}\left(-\frac{e^{-sA}}{s} + \frac{1}{s}\right) \\ &= \frac{1}{s}, s > 0\end{aligned}$$

- $\mathcal{L}\left\{e^{(a+ib)t}\right\} = \int_0^\infty e^{-st}e^{(a+ib)t}\,\mathrm{d}t$
$$= \left[\frac{1}{a-s+ib}e^{((a-s)+ib)t}\right]_0^\infty$$
$$= \lim_{A\to\infty}\left(\frac{e^{((a-s)+ib)A}}{a-s+ib} - \frac{1}{a-s+ib}\right)$$
$$= \frac{1}{s-(a+ib)}, s > a$$
- $\mathcal{L}\left\{\sin t\right\} = \mathcal{L}\left\{\frac{e^{it}-e^{-it}}{2i}\right\}$
$$= \frac{1}{2i}\mathcal{L}\left\{e^{it}\right\} - \frac{1}{2i}\mathcal{L}\left\{e^{-it}\right\}$$
$$= -\frac{1}{2i}\frac{1}{-s+i} + \frac{1}{2i}\frac{1}{-s-i}$$
$$= -\frac{1}{2i}\frac{-s-i-(-s+i)}{(-s)^2-i^2}$$
$$= -\frac{1}{2i}\frac{-2i}{s^2+1}$$
$$= \frac{1}{s^2+1}$$
- In reality we just look these up from a table

Linearity of the Laplace Transform

Theorem

The Laplace transform is linear:

$$\mathcal{L}\left\{c_1f_1(t) + c_2f_2(t)\right\} = c_1\mathcal{L}\left\{f_1(t)\right\} + c_2\mathcal{L}\left\{f_2(t)\right\}$$

If $\mathcal{L}\left\{f_1\right\}$ exists for $t > s_1$ and $\mathcal{L}\left\{f_2\right\}$ exists for $t > s_2$ then the linear combination exists for $t > \max(s_1, s_2)$

Lecture 24, Nov 3, 2022

Definition of the Laplace Transform (again)

- So far the solutions to first order and second order ODEs all seem to contain exponentials and sinusoids
- Can we transform these solutions to something nicer?
- The Fourier transform: $\mathcal{F}\left\{f\right\}(\omega) = \int_{-\infty}^\infty e^{-i\omega t}f(t)\,\mathrm{d}t$
  - This takes you from the time domain to the frequency domain
  - Can be though of as a dot product between the function of various sines and cosines of various frequencies
  - If we have a sine or a cosine, we get a very simple representation when taken to the frequency domain
  - However the Fourier transform doesn't work well with exponentials, which is why we need the Laplace transform
- The Laplace transform: $\mathcal{L}\left\{f\right\}(s) = \int_0^\infty e^{-st}f(t)\,\mathrm{d}t$
  - Note $\mathcal{L}\left\{f\right\}(\sigma + i\omega) = \int_0^\infty e^{-(\sigma+i\omega)t}f(t)\,\mathrm{d}t$

    * When $\sigma = 0$, we get the Fourier transform
  - The Fourier transform is a slice of the Laplace transform
  - The Fourier transform can only handle sines and cosines, which are pure oscillations that do not decay or grow, whereas with the Laplace transform we can also handle decaying exponentials
    * This makes it suitable for functions that appear in ODEs

Existence of the Laplace Transform

- $f(t)$ needs to be piecewise continuous for the integration to be possible
- $f(t)$ must not dominate the e^{-st}, otherwise the integral will diverge
- If $|f(t)| \leq Ke^{at}$ for $t < M$, then the Laplace transform exists if and only if $a < s$
  - We can show this using the limit comparison test

Theorem

The Laplace transform $\mathcal{L}\{f\}(s)$ exists for $s > a$ if:
1. f is piecewise continous on $0 \leq t \leq A$ for any positive A
2. f is of exponential order so that $|f(t)| \leq Ke^{at}$ when $t \geq M$

- Proof: $\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st}f(t)\,\mathrm{d}t = \int_0^M e^{-st}f(t)\,\mathrm{d}t + \int_M^\infty e^{-st}f(t)\,\mathrm{d}t$
  - The first part exists by hypothesis 1, the second exists by hypothesis 2

Lecture 25, Nov 4, 2022

Properties of the Laplace Transform

1. Exponential in t is a shift in s: $\mathcal{L}\{f(t)\} = F(s), s > a \implies \mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-c), s > a + c$
   - e.g. $\mathcal{L}\left\{e^{-2t}\sin(4t)\right\} = \dfrac{4}{(s+2)^2 + 16}, s > 2$
2. Derivative in t is a multiplication by s: $\mathcal{L}\{f'(s)\} = s\mathcal{L}\{f(t)\} - f(0)$
   - Derived by integration by parts
   - $$\begin{aligned}\int_0^\infty e^{-st}f'(t)\,\mathrm{d}t &= \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)\,\mathrm{d}t \\ &= s\mathcal{L}\{f(t)\} - f(0)\end{aligned}$$
   - We need to assume that the function does not blow up
3. Corollary: $\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
   - e.g. $\dfrac{\mathrm{d}^4y}{\mathrm{d}t^4} - y = 0$ with initial conditions $y(0) = 0$, $\left.\dfrac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=0} = 0$, $\left.\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}\right|_{t=0} = 0$, $\left.\dfrac{\mathrm{d}^3y}{\mathrm{d}t^3}\right|_{t=0} = 1$
     - Take the Laplace transform of both sides
     - $s^4\mathcal{L}\{y(t)\} - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0) - \mathcal{L}\{y\} = 0$
     - $s^4\mathcal{L}\{y(t)\} - \mathcal{L}\{y\} = 1 \implies \mathcal{L}\{y(t)\} = \dfrac{1}{s^4 - 1}$
   - Starting with an ODE in t, we get a polynomial in s
4. Multiplication by t^n is an n-th derivative in s: $\mathcal{L}\{t^nf(t)\} = (-1)^nF^{(n)}(s)$

Lecture 26, Nov 14, 2022

Solving ODEs With Laplace Transforms

- Apply the Laplace transform to both sides of an ODE and we can turn the differential equation into an algebraic equation in the Laplace domain

- Instead of solving the ODE in time domain directly, we apply the Laplace transform, solve the algebraic equation in the Laplace domain, and then use the inverse Laplace transform to turn it back into the time domain
- When we rearrange for $Y(s)$ we usually get a rational function, the ratio of one polynomial to another

Inverse Laplace Transform

- $f(t) = \mathcal{L}^{-1}\{F\}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)\,\mathrm{d}s$
  - Problem: We don't know how to do this integral!

Theorem

Given two piecewise continuous, exponential order functions $f(t), g(t)$ such that

$$\mathcal{L}\{f\} = \mathcal{L}\{g\}$$

Then $f(t) = g(t)$ at all points where both f and g are continuous

- This theorem guarantees us that when we apply the inverse transform, we actually get the solution provided we have continuity
- Note the inverse Laplace transform is linear like the forward transform
- Example: $\frac{s+1}{s^2+2s+5}$
  - Complete the square: $\frac{s+1}{(s+1)^2+4}$
  - Note $e^{at}\cos(bt) = \frac{s-a}{(s-a)^2+b^2}$
  - Using the table, $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\} = e^{-t}\cos(2t)$
- Often we get something that's not in the table; in general if we get $F(s) = \frac{P(s)}{Q(s)}$, then we need to do partial fractions
- Let $Q(s) = (s-s_1)(s-s_2)\cdots(s-s_n)$ then we can rewrite $F(s) = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \cdots + \frac{a_n}{s-s_n}$
  - With repeated roots, if $Q(s) = (s-s_0)^k$ then $F(s) = \frac{a_1}{s-s_0} + \frac{a_2}{(s-s_0)^2} + \cdots + \frac{a_k}{(s-s_0)^k}$
  - With complex roots, $Q(s) = (s-(\mu+i\nu))^k(s-(\mu-i\nu))^k$ then $F(s) = \frac{a_1(s-\mu)+b_1\nu}{(s-\mu)^2+\nu^2} + \cdots + \frac{a_k(s-\mu)+b_k\nu}{((s-\mu)^2+\nu^2)^k}$
- Example: $\frac{s-2}{s^2-4s-5}$
  - $Q = (s-5)(s+1)$
  - $\frac{s-2}{(s-5)(s+1)} = \frac{A}{(s-5)} + \frac{B}{(s+1)}$
  - $s-2 = (s+1)A + (s-5)B \implies s-2 = s(A+B) + (A-5B) \implies \begin{cases} A+B=1 \\ A-5B=-2 \end{cases}$
  - $A = \frac{1}{2}, B = \frac{1}{2}$
  - $\frac{s-2}{s^2-4s-5} = \frac{1}{2(s-5)} + \frac{1}{2(s+1)}$
  - From the table, $\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-4s-5}\right\} = \frac{1}{2}e^{5t} + \frac{1}{2}e^{-t}$

Lecture 27, Nov 17, 2022

Unit Step Function (Heaviside Function)

- Heaviside step function (aka indicator function): $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$
- Translated step function: $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$
- Indicator step function: $u_{cd}(t) = u_c(t) - u_d(t) = \begin{cases} 0 & t < c, t \geq d \\ 1 & c \leq t < d \end{cases}$
- From the step function we can construct other functions, e.g. a triangular pulse is $(-1+t)u_{12}(t) + (3-t)u_{23}(t)$

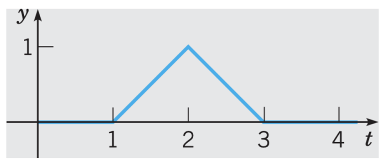

Figure 20: Triangular pulse

Laplace Transform of the Step Function

- If $\mathcal{L}\left\{f(t)\right\} = F(s), s > a \geq 0$, then $\mathcal{L}\left\{u_c(t)f(t-c)\right\} = e^{-cs}F(s), s > a$
  - This is the dual of $\mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-c)$
- An exponential in the time domain is a shift in the s domain; an exponential in the s domain is also a shift in the time domain
- $\mathcal{L}\left\{u(t)\right\} = \frac{1}{s}, s > 0$
- $\mathcal{L}\left\{u_c(t)\right\} = \frac{e^{-cs}}{s}, s > 0$
- $\mathcal{L}\left\{u_{cd}(t)\right\} = \mathcal{L}\left\{u_c(t)\right\} - \mathcal{L}\left\{u_d(t)\right\} = \frac{e^{-cs} - e^{-ds}}{s}, s > 0$

Periodic Functions

Definition

A function f is periodic if

$$f(t+T) = f(t)$$

where T is the period

- The window function: $f_T(t) = f(t)(1 - u_T(t)) = \begin{cases} f(t) & t \leq T \\ 0 & \text{otherwise} \end{cases}$
- We can use it to construct periodic functions as $f(t) = \sum_{n=0}^{\infty} f_T(t - nT)u_{nT}(t)$
- Using this, we can Laplace transform any periodic function

- $\mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty} \mathcal{L}\{f_T(t - nT)u_{nT}(t)\}$
$$= \sum_{n=0}^{\infty} e^{-nTs}\mathcal{L}\{f_T(t)\}$$
$$= \mathcal{L}\{f_T(t)\}\frac{1}{1 - e^{-Ts}}$$

Theorem

If f is periodic with period T and is piecewise continuous on $[0, T]$, then

$$\mathcal{L}\{f(t)\} = \frac{F_T(s)}{1 - e^{-Ts}}$$

where $F_T(s) = \mathcal{L}\{f_T(t)\} = \mathcal{L}\{f(t)(1 - u_T(t))\}$

Lecture 28, Nov 18, 2022

Differential Equations With Discontinuous Forcing Functions

- Example: flipping a switch, or turning a knob are all examples of discontinuous forcing functions
- Example 1: $y'' + 4y = g(t), y(0) = 0, y'(0) = 0, g(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{t-5}{5} & 5 \leq t < 10 \\ 1 & t \geq 10 \end{cases}$
  - This forcing function is known as a *ramp*
  - First express $g(t)$ in terms of step functions: $g(t) = \frac{t-5}{5}u_5(t) - \frac{t-10}{5}u_{10}(t)$
  - $\mathcal{L}\{g\} = \frac{1}{5}\mathcal{L}\{(t-5)u_5(t) - (t-10)u_{10}(t)\} = \frac{1}{5s^2}(e^{-5s} - e^{-10s})$
  - $\mathcal{L}\{y'' + 4y\} = s^2Y(s) + 4Y(s) = \mathcal{L}\{g\}$
  - $Y(s) = (e^{-5s} - e^{-10s})\frac{1}{5s^2(s^2+4)}$
  - Let $H(s) = \frac{1}{s^2(s^2+4)}$, then $y(t) = \frac{u_5(t)h(t-5) - u_{10}(t)h(t-10)}{5}$ where $h(t) = \mathcal{L}^{-1}\{H(s)\}$
  - By partial fractions $H(s) = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4} \implies h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$
- Example 2: $y'' + \pi^2 y = f(t), y(0) = 0, y'(0) = 0$ where $f(t)$ is a square wave
  - Use the periodic function Laplace transform formula
  - We need a window function f_2 which we could construct as $f_2 = u_0(t) - u_1(t)$
  - $F_2(s) = \frac{1}{s}(1 - e^{-s})$
  - From the previous lecture $F(s) = \frac{F_2(s)}{1 - e^{-st}} = \frac{1 - e^{-s}}{s(1 - e^{-2s})} = \frac{1 - e^{-s}}{s(1 - e^{-2s})(1 + e^{-2s})} = \frac{1}{s(1 + e^{-s})}$
  - $\mathcal{L}\{y'' + \pi^2 y\} = (s^2 + \pi^2)Y(s) = F(s) \implies Y(s) = \frac{1}{s(1+e^{-s})(s^2+\pi^2)} = \frac{1}{s(s^2+\pi^2)}\frac{1}{1+e^{-s}}$
  - Let $H(s) = \frac{1}{s(s^2+\pi^2)}$
  - $Y(s) = \sum_{k=1}^{\infty}(-1)^k e^{-ks}H(s)$
  - By partial fractions $h(t) = \frac{1}{\pi^2}(1 - \cos(\pi t))$

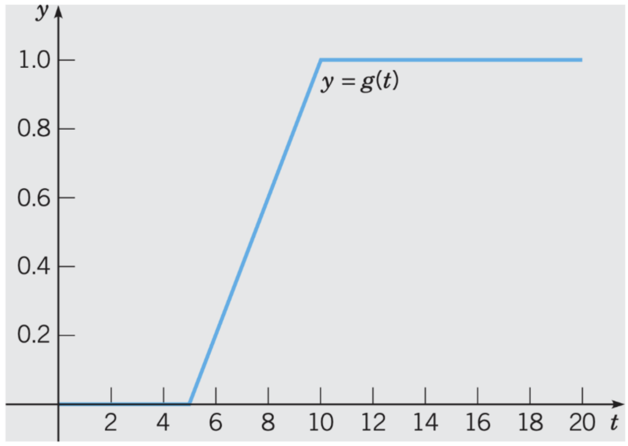

Figure 21: Ramp forcing function

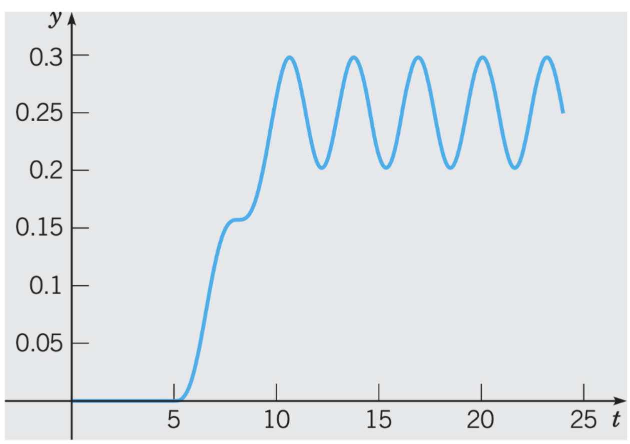

Figure 22: Example 1 solution

- Therefore $y(t) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{\pi^2}(1-\cos(\pi(t-k)))u_k(t)$

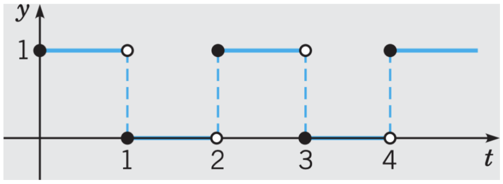

Figure 23: Square wave forcing function

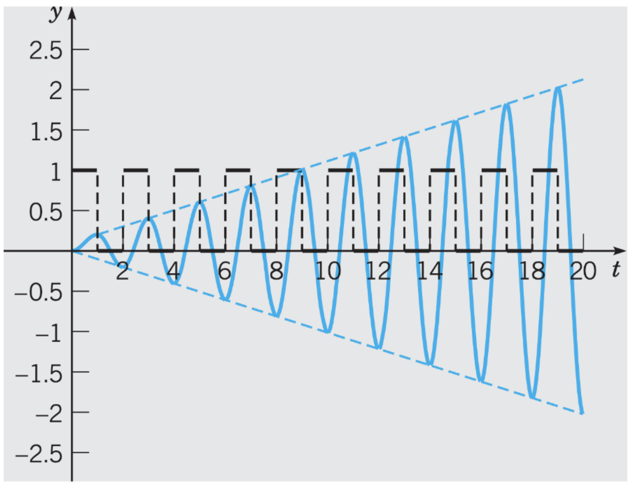

Figure 24: Example 2 solution

Lecture 29, Nov 21, 2022

Impulse Functions

- For some number ϵ we can define a delta epsilon function $\delta_\epsilon(t) = \dfrac{u_0(t) - u_\epsilon(t)}{\epsilon} = \begin{cases} \dfrac{1}{\epsilon} & 0 \le t \le \epsilon \\ 0 & \text{otherwise} \end{cases}$
- The Dirac delta function is $\lim_{\epsilon \to 0} \delta_\epsilon(t)$
  - At a single point, the value is infinite
  - The integral over the peak is 1
- $\mathcal{L}\left\{\delta_\epsilon(t)\right\} = \mathcal{L}\left\{\dfrac{u_0(t) - u_\epsilon(t)}{\epsilon}\right\} = \dfrac{1 - e^{-\epsilon s}}{\epsilon s}$
- Consider $y'' + y = I_0\delta_\epsilon(t), y(0) = 0, y'(0) = 0$
  - $\mathcal{L}\left\{y'' + y'\right\} = (s^2+1)Y(s)$
  - $Y(s) = \dfrac{I_0}{\epsilon}(1 - e^{-\epsilon s})\dfrac{1}{s(s^2+1)}$
  - Let $H(s) = \dfrac{1}{s(s^2+1)} \implies h(t) = 1 - \cos(t)$
  - $Y(s) = \dfrac{I_0}{\epsilon}(H(s) - e^{-\epsilon s}H(s)) \implies y(t) = \dfrac{I_0}{\epsilon}\left(u_0(t)(1-\cos(t)) - u_\epsilon(t)(1 - \cos(t-\epsilon))\right)$

- $y_\epsilon(t) = \begin{cases} 0 & t \leq 0 \\ \dfrac{I_0}{\epsilon}(1 - \cos(t)) & 0 \leq t \leq \epsilon \\ \dfrac{I_0\left(\cos(t-\epsilon) - \cos(t)\right)}{\epsilon} & t > \epsilon \end{cases}$
- $\lim_{\epsilon \to 0} y_\epsilon(t) = y(t) = \begin{cases} 0 & t \leq 0 \\ -I_0 \dfrac{\mathrm{d}}{\mathrm{d}t} \cos(t) & t > 0 \end{cases} = \begin{cases} 0 & t \leq 0 \\ I_0 \sin(t) & t > 0 \end{cases}$
- $y(t) = I_0 u(t) \sin(t)$

The Dirac Delta Function

Definition

The Dirac delta function is the function $\delta(t)$ with the following properties:

- $\delta(t - t_0) = 0$ whenever $t \neq t_0$
- $\displaystyle\int_a^b \delta(t - t_0)\,\mathrm{d}t = \begin{cases} 1 & a \leq t_0 \leq b \\ 0 & \text{otherwise} \end{cases}$
- $\displaystyle\int_a^b f(t)\delta(t - t_0)\,\mathrm{d}t = f(t_0)$

- Using the sifting property, we have $\mathcal{L}\left\{\delta(t - t_0)\right\} = e^{-st_0}$ and so $\mathcal{L}\left\{\delta(t)\right\} = 1$

Lecture 30, Nov 28, 2022

Convolutions

Definition

The convolution of two functions $f(t)$ and $g(t)$ is

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau)\,\mathrm{d}\tau$$

The discrete version for two sequences $f[n]$ and $g[n]$ is

$$(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m]g[n - m]$$

- Taking one function, flipping it around, shifting it by some amount, and seeing how much the two functions correlate
- Properties:
  - Commutativity: $f * g = g * f$
  - Distributivity: $f * (g_1 + g_2) = f * g_1 + f * g_2$
  - Associativity: $(f * g) * h = f * (g * h)$
  - Zero: $f * 0 = 0 * f = 0$
- Convolution in time domain is multiplication in s domain: $\mathcal{L}\left\{f * g\right\} = \mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\}$ (Convolution Theorem)
  - This also holds for the Fourier transform
  - Now if we have to take the inverse Laplace transform of some product, we can just use a convolution!

Input-Output Problem

- Consider the ODE $ay'' + by' + cy = g(t), y(0) = y_0, y'(0) = y_1$
  - Laplace transform: $(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_1 = G(s)$
  - $Y(s) = \dfrac{(as + b)y_0 + ay_1 + G(s)}{as^2 + bs + c}$
  - Let $H(s) = \dfrac{1}{as^2 + bs + c}$ be the *transfer function* of this system
  - $Y(s) = H(s)((as + b)y_0 + ay_1) + H(s)G(s)$
- We get the solution $y(t) = \mathcal{L}^{-1}\left\{H(s)((as + b)y_0 + ay_1)\right\} + \int_0^t h(t-\tau)g(\tau)\,\mathrm{d}\tau$
  - The first part of this, $\mathcal{L}^{-1}\left\{H(s)((as + b)y_0 + ay_1)\right\}$, is the *free response*
    * This is the response of the system to the initial conditions $y(0)$ and $y'(0)$ only, disregarding the forcing function
    * This can be thought of as the solution to the homogeneous system
  - The second part, $\int_0^t h(t-\tau)g(\tau)\,\mathrm{d}\tau$, is the *forced response*
    * This is the response of the system to $g(t)$ only, without any initial conditions (i.e. $y(0) = y'(0) = 0$)
    * This can be thought of as a particular solution to the non-homogeneous system
  - Combining the two we get the total response $y(t)$
- Note the forced response is $H(s)G(s)$; if we take $g(t) = \delta(t)$, then $H(s)G(s) = H(s)\mathcal{L}\left\{\delta(t)\right\} = H(s)$
  - The transfer function $H(s)$ is simply the impulse response of $ay''+by'+cy = \delta(t), y(0) = 0, y'(0) = 0$
  - Knowing the impulse response of the system allows us to easily determine the forced response

Lecture 31, Dec 1, 2022

Example

- System: $y'' + 2y' + 5y = g(t)$
- Transfer function: $H(s) = \dfrac{1}{s^2 + 2s + 5} = \dfrac{1}{(s+1)^2 + 4}$
- Impulse response: $h(s) = \mathcal{L}^{-1}\left\{H(s)\right\} = \dfrac{1}{2}e^{-t}\sin(2t)$
- Homogeneous solution: $\lambda = -1 \pm 2i \implies y_g(t) = e^{-t}(c_1\cos(2t) + c_2\sin(2t))$
- Particular solution (forced response): $h * g = \int_0^t \dfrac{1}{2}e^{-(t-\tau)}\sin(2(t-\tau))g(\tau)\,\mathrm{d}\tau$
- General solution: $y(t) = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t) + \int_0^t \dfrac{1}{2}e^{-(t-\tau)}\sin(2(t-\tau))g(\tau)\,\mathrm{d}\tau$

Partial Differential Equations (PDEs)

- Examples:
  - Heat equation $\dfrac{\partial u}{\partial t} = D\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$
    * Heat dispersing in an object
  - Wave equation: $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$
  - Laplace equation: $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$
- PDEs have more than one variable, whereas ODEs only have one variable x or t
- Consider 1D heat conduction along a cylinder, with temperature $u(x,t)$
- PDE: $u_t = a^2u_{xx}$, for $0 < x < L, t > 0$
  - α is the thermal diffusivity
  - Intuition: Points that are more "concentrated" in heat will have that heat spread out faster

- Initial conditions: $u(x,0) = f(x), 0 \le x \le L$
    - Initial temperature distribution
  - Boundary conditions: $u(0,t) = 0, u(L,t) = 0, t > 0$
    - Temperature at both ends of the rod is zero
- How would we approach something like this?
- Separation of variables: guess solution $u(x,t) = X(t)T(t)$
  - Substitution into ODE: $\frac{X''}{x} = \frac{1}{\alpha^2}\frac{T'}{T}$
  - Notice the left hand side is only a function of x while the right hand side is only a function of t, so for both to equal each other for all x and t, both must be constants
  - $\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda$
  - This gives us two ODEs: $\begin{cases} X'' + \lambda X = 0 \\ T' + \alpha^2 \lambda T = 0 \end{cases}$

Lecture 32, Dec 2, 2022

Solving the Heat Equation

- We separated the system: $\begin{cases} X'' + \lambda X = 0 \\ T' + \alpha^2 \lambda T = 0 \end{cases}$
- In X: $X'' + \lambda X = 0$ with boundary conditions $X(0) = 0, X(L) = 0$
  - This gives us the general solution $X(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$
  - Using boundary conditions we have $c_1 = 0, c_2 \sin(\sqrt{\lambda} L) = 0$
  - This gives us solutions of $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ for integers n, giving $\lambda_n = \frac{n^2\pi^2}{L^2}$
  - Note $X(x)$ is the eigenfunction and λ is the eigenvalue of the $\frac{\partial^2}{\partial x^2}$ operator
- Substituting λ into the time ODE, we get $T(t) \propto \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)$
- This gives us the set of fundamental solutions $u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}$
- The general solution is $u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}$
- With the initial condition $f(x) = u(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)$
- Now we can get c_n as the Fourier coefficients: $c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \,\mathrm{d}x$

Lecture 33, Dec 5, 2022

Piecewise Continuous Functions as Vectors

- The set of piecewise continuous functions on (a,b) is denoted by $\mathrm{PC}[a,b]$
- This set is closed under scalar multiplication and addition, so it forms a vector space
- We define the *inner product* of two members of $\mathrm{PC}[a,b]$ as $\langle f, g\rangle = \int_a^b f(x)g(x)\,\mathrm{d}x$
  - The inner product is like a more generalized dot product
  - The usual properties of dot products are also satisfied: commutativity, linearity, distributivity, and $\langle f, f\rangle = 0$ if and only if $f = 0$
- Using the inner product we can define the *norm* as $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b (f(x))^2\,\mathrm{d}x}$

- $f, g \in \text{PC}[a,b]$ are *orthogonal* if $\langle f, g \rangle = \int_a^b f(x)g(x)\,\mathrm{d}x = 0$

Definition

A set of functions

$$S = \{\, \phi_1(x), \phi_2(x), \cdots \,\} \in \text{PC}[a,b]$$

are orthogonal if

$$\langle \phi_n, \phi_m \rangle = 0, n \neq m$$

and orthonormal if

$$\|\phi_n\| = 1, \forall n$$

i.e. $\langle \phi_n, \phi_m \rangle = \delta_{mn}$

Fourier's Theorem

- An important orthonormal set on $\text{PC}[-L, L]$ is $\left\{ \sqrt{\frac{2}{L}}\frac{1}{2}, \sqrt{\frac{1}{L}}\sin\left(\frac{m\pi x}{L}\right), \sqrt{\frac{1}{L}}\cos\left(\frac{m\pi x}{L}\right), \cdots : m \in \mathbb{N} \right\}$
  - $\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,\mathrm{d}x = \delta_{mn}\pi$

Theorem

Fourier's Theorem: suppose f is periodic with period $2L$, $f, f' \in \text{PC}[-L, L]$, then f can be expressed as a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)$$

with Fourier coefficients given by

$$a_0 = \frac{1}{L}\langle f(x), 1 \rangle \tag{1}$$

$$a_m = \frac{1}{L}\left\langle f(x), \cos\left(\frac{m\pi x}{L}\right) \right\rangle \tag{2}$$

$$b_m = \frac{1}{L}\left\langle f(x), \sin\left(\frac{m\pi x}{L}\right) \right\rangle \tag{3}$$

- The Fourier theorem is the direct analogue of the fact that you can represent a vector in another basis by taking the dot product of the vector with each of the basis vectors if the basis is orthonormal
  - A Fourier transform is nothing but a change of basis
- In the case of the discrete Fourier transform, $Ff = \hat{f}$, and due to orthonormality $F^T\hat{f} = f$