Lecture 5, Sep 19, 2022

Boolean Algebra Basics

- Canonical SOP and POS representations can be large and inefficient; boolean algebra lets us simplify and optimize them
- Axioms of Boolean Algebra:
  1. $0 \cdot 0 = 0$
  2. $1 \cdot 1 = 1$
  3. $0 \cdot 1 = 1 \cdot 0 = 0$
  4. $x = 0 \implies \bar{x} = 1$
- Duality: given a logic expression, swapping all 0 with 1 and $\cdot$ with $+$ leaves the expression still valid; this gives every axiom a *dual form*:
  1. $1 + 1 = 1$
  2. $0 + 0 = 0$
  3. $1 + 0 = 0 + 1 = 1$
  4. $x = 1 \implies \bar{x} = 0$
- Derived rules: ($\forall x$:)
  5. $x \cdot 0 = 0$, dual: $x + 1 = 1$
  6. $x \cdot 1 = x$, dual: $x + 0 = x$
  7. $x \cdot x = x$, dual: $x + x = x$
  8. $x \cdot \bar{x} = 0$, dual: $x + \bar{x} = 1$
  9. $\bar{\bar{x}} = x$
- Derived identities: ($\forall x, y, z$:)
  10. Commutativity: $x \cdot y = y \cdot x$, dual: $x + y = y + x$
  11. Associativity: $x(yz) = (xy)z$, dual: $x + (y + z) = (x + y) + z$
  12. Distributivity: $x(y + z) = xy + xz$, dual: $x + (yz) = (x + y)(x + z)$
  13. Absorption: $x + xy = x$, dual: $x(x + y) = x$
  14. Combination: $xy + x\bar{y} = x$, dual: $(x + y)(x + \bar{y}) = x$
  15. DeMorgan's Theorem: $\overline{xy} = \bar{x} + \bar{y}$, dual: $\overline{(x + y)} = \bar{x}\bar{y}$
      - Proof: $\overline{xy} = \bar{x}\bar{y} + \bar{x}y + x\bar{y}$ Canonical SOP
        $= \bar{x}\bar{y} + \bar{x}y + \bar{x}\bar{y} + x\bar{y}$ Rule 7
        $= \bar{x} + \bar{y}$ Combination
  16. $x + \bar{x}y = x + y$, dual: $x(\bar{x} + y) = xy$

Proof by Perfect Induction

- Proving a statement by enumerating all the possible cases
- Example: proving $x + (yz) = (x + y)(x + z)$

x	y	z	yz	$x + (yz)$	$x + y$	$x + z$	$(x + y)(x + z)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1