Lecture 29, Nov 22, 2022

Semi-Infinite Solids

- Consider an object with surface temperature T_s and internal temperature T_s
  - The skin layer is the outer layer of the solid where the temperature is a gradient; heat transfer is meaningfully occurring
  - The core is the part that's relatively untouched by heat transfer so it has a roughly constant temperature
  - The actual temperature distribution would be an exponential, and the skin layer is the region where the exponential is changing fast, whereas the core is the asymptote
  - The dividing point is relatively subjective
- How does the skin depth δ vary with time?
- Apply a semi-quantitative scaling analysis, with the goal of finding functional relationships
  - Starting with the conduction equation: $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$
  - $\dfrac{\partial^2 T}{\partial x^2} \sim \dfrac{\frac{\partial T}{\partial x}\big|_{x=\delta} - \frac{\partial T}{\partial x}\big|_{x=0}}{\delta}$
  - $\dfrac{\partial T}{\partial x}\bigg|_{x=\delta} = 0$ since at that point the heat transfer is done, so temperature is not changing much
  - $\dfrac{\partial T}{\partial x}\bigg|_{x=0} \approx \dfrac{T_i - T_s}{\delta}$ is the slope roughly at the surface
  - $\dfrac{\partial^2 T}{\partial x^2} \sim \dfrac{0 - \frac{T_i - T_s}{\delta}}{\delta} = \dfrac{T_s - T_i}{\delta^2}$
  - $\dfrac{\partial T}{\partial t} \sim \dfrac{T_s - T_i}{\Delta t} = \dfrac{T_s - T_i}{t}$
  - Substituting: $\dfrac{T_s - T_i}{\delta^2} \approx \dfrac{1}{\alpha}\dfrac{T_s - T_i}{t} \implies \delta(t) \sim \sqrt{\alpha t}$
- δ scales with $\sqrt{\alpha t}$
  - This is not an exact equivalence, but now we know roughly how deep the heat transfer gets as time goes on
- Consider a sphere with radius r_0, then heat transfer reaches the centre when $\delta = r_0$; so we can devise a characteristic time $t_c = \dfrac{r_0^2}{\alpha}$ for the heat transfer to reach the centre
  - If $t \ll t_c$ then we can treat the body as *semi-infinite*, i.e. infinite in one direction
- With a semi-infinite assumption we have an exact solution to the transient heat conduction problem
  - $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$ with boundary conditions $T(0,t) = T_s, T(\infty, t) = T_i, T(x, 0) = T_i$
  - Using the scaling analysis to relate t and x
    * $\delta(t) \sim \sqrt{\alpha t}$
    * Define the similarity variable $\eta = \dfrac{x}{2\delta} = \dfrac{x}{2\sqrt{\alpha t}}$
    * $\dfrac{\partial T}{\partial t} = \dfrac{\mathrm{d}T}{\mathrm{d}\eta}\dfrac{\partial \eta}{\partial t} = \dfrac{\mathrm{d}T}{\mathrm{d}\eta}\left(\dfrac{-x}{4t\sqrt{\alpha t}}\right)$
    * $\dfrac{\partial T}{\partial x} = \dfrac{\mathrm{d}T}{\mathrm{d}\eta}\dfrac{\partial \eta}{\partial x} = \dfrac{\mathrm{d}T}{\mathrm{d}\eta}\left(\dfrac{1}{2\sqrt{\alpha t}}\right)$
    * $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{\mathrm{d}}{\mathrm{d}\eta}\left(\dfrac{\mathrm{d}T}{\mathrm{d}\eta}\right)\left(\dfrac{\partial \eta}{\partial x}\right)^2 = \dfrac{\mathrm{d}^2 T}{\mathrm{d}x^2}\left(\dfrac{1}{4\alpha t}\right)$
    * $\dfrac{1}{4\alpha t}\dfrac{\mathrm{d}^2 T}{\mathrm{d}\eta^2} = \dfrac{1}{\alpha}\left(-\dfrac{x}{4t\sqrt{\alpha t}}\right)\dfrac{\mathrm{d}T}{\mathrm{d}\eta}$
    * $\dfrac{\mathrm{d}^2 T}{\mathrm{d}\eta^2} = -\dfrac{x}{\sqrt{\alpha t}}\dfrac{\mathrm{d}T}{\mathrm{d}\eta} = -2\eta\dfrac{\mathrm{d}T}{\mathrm{d}\eta}$
  - New boundary conditions: $T(0) = T_s, T(\infty) = T_i$
  - Let $w = \dfrac{\mathrm{d}T}{\mathrm{d}\eta} \implies \dfrac{\mathrm{d}w}{\mathrm{d}\eta} = -2\eta w$

- Solve: $\ln w = -\eta^2 + C \implies w = c_0 e^{-\eta^2} = \frac{\mathrm{d}T}{\mathrm{d}\eta}$
- $T = c_0 \int_0^\eta e^{-u^2}\,\mathrm{d}u + c_1$
- Boundary conditions: $T(0) = c_1 = T_s, T(\infty) = c_0\frac{\sqrt{\pi}}{2} + T_s \implies c_0 = \frac{2(T_i - T_s)}{\sqrt{\pi}}$
- $\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}\,\mathrm{d}u = \mathrm{erf}(\eta)$
- $1 - \frac{T - T_s}{T_i - T_s} = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}(\eta)$
  * erfc is the complementary error function
- $\frac{T - T_i}{T_s - T_i} = \mathrm{erfc}(\eta)$

• For heat flux: $\dot{q}_s = -h\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=0} = -h\left.\frac{\mathrm{d}T}{\mathrm{d}\eta}\frac{\partial\eta}{\partial x}\right|_{\eta=0}$
  - Differentiating the temperature profile we get $\frac{1}{T_i - T_s}\frac{\mathrm{d}T}{\mathrm{d}\eta} = \frac{2}{\sqrt{\pi}}e^{-\eta^2} \implies \frac{\mathrm{d}T}{\mathrm{d}\eta} = \frac{2(T_i - T_s)}{\sqrt{\pi}}e^{-\eta^2}$
  - $\frac{\partial\eta}{\partial x} = \frac{1}{2\sqrt{\alpha t}}$
  - Plugging these in $\dot{q} = -k(T_i - T_s)\cdot\frac{2}{\sqrt{\pi}}\cdot\frac{1}{2\sqrt{\alpha t}}$
  - Simplify to get a heat flux at the base of $\dot{q} = \frac{k(T_s - T_i)}{\sqrt{\pi\alpha t}}$

• Contact of two semi-infinite bodies: joining together two semi-infinite materials A and B, applying the same analysis as before
  - We have $T_{s,A} = T_{s,B} = T_s$ and $\dot{q}_{s,A} = \dot{q}_{s,B}$
  - $-\frac{k(T_s - T_{A,i})}{\sqrt{\pi\alpha_A t}} = \frac{k_B(T_s - T_{B,i}}{\sqrt{\pi\alpha_B t}}$
  - $\frac{T_{A,i} - T_s}{T_s - T_{B,i}} = \frac{\sqrt{(k\rho c)_B}}{\sqrt{(k\rho c)_A}} = \frac{\gamma_B}{\gamma_A}$
    * γs are known as the *effusivities*
  - $T_s = \frac{\gamma_A T_{A,i} + \gamma_B T_{B,i}}{\gamma_A + \gamma B}$
    * Notice this is constant

Summary

When the time scale is such that the skin depth $\delta = \sqrt{\alpha t} \ll L$ where L is the characteristic length, we can treat a solid as semi-infinite, in which case

$$\frac{T - T_i}{T_s - T_i} = \mathrm{erfc}(\eta) = 1 - \mathrm{erf}(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}\,\mathrm{d}u$$

where $\eta = \frac{x}{2\sqrt{\alpha t}}$ is the similarity variable; this results in a heat transfer at the base of

$$\dot{q} = \frac{k(T_s - T_i)}{\sqrt{\pi\alpha t}}$$

This usually applies in cases of very low Bi, i.e. $R_{cond} \gg R_{conv}$