Lecture 28, Nov 21, 2022

Transient Conduction (Non-Lumped)

- When $B_i > 0.1$ we can no longer neglect the temperature difference in the body, so we have to solve the complete heat conduction equation
- When we insert a plane wall into a fluid, we should see a bump in the temperature in the middle of the wall that becomes flatter over time
- In this case $T = T(x, t)$ if we assume 1D conduction
- $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
  - $\alpha = \frac{k}{\rho c_p}$ is the thermal diffusivity
- Initial conditions: $T(x, 0) = T_i$, boundary conditions: at L, $-k\frac{\partial T}{\partial x} = h(T(L, t) - T_\infty)$
  - Use symmetry, $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$ (note zero is the centre of the plate)
- Use a change of variables $X = \frac{x}{L}, \mathrm{Bi} = \frac{hL}{k}, \tau = \frac{\alpha t}{L^2}$
  - $\tau = \mathrm{Fo}$ is also known as the Fourier number, a unitless measure of time (note this is a variable)
    * The Fourier number is the ratio of conductive heat transfer to energy increase in the system
    * Imagine a cube with sides L with conductive heat transfer in, convective heat transfer out
    * From Fourier's law $\dot{Q}_{cond} = kA\frac{\mathrm{d}T}{\mathrm{d}x} = kL^2\frac{\Delta T}{L} = kL\Delta T$
    * $\Delta E = mc\frac{\mathrm{d}T}{\mathrm{d}t} = \rho L^3 c\frac{\Delta T}{t}$
    * $\frac{\dot{Q}_{cond}}{\Delta E} = \frac{k}{\rho c}\frac{t}{L^2} = \frac{\alpha t}{L^2} = \mathrm{Fo}$
  - $\theta(X, \mathrm{Fo}) = \frac{T - T_\infty}{T_i - T_\infty}$ is the normalized thermal driving force
- $\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \mathrm{Fo}}$
  - Boundary conditions $\left.\frac{\partial \theta}{\partial X}\right|_{X=0} = 0$
  - At the walls $\left.\frac{\partial \theta}{\partial X}\right|_{X=1} = -\mathrm{Bi}\theta(1, \mathrm{Fo})$
- This can be solved analytically, but we're not going to do so
- In the simplest case of the lumped capacitance $\theta = e^{-\frac{hA}{\rho V c}t} = e^{-\left(\frac{hL}{k}\right)\left(\frac{\alpha t}{L^2}\right)t} = e^{-\mathrm{BiFo}}$
- In more complex cases $\theta = \theta(X, \mathrm{Bi}, \mathrm{Fo})$ is a complicated function, but the textbook gives some approximations for different geometries for sufficiently large t
  - For the plane wall: $\theta = \frac{T(x, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \mathrm{Fo}} \cos\left(\lambda_1 \frac{x}{L}\right)$
    * A_1, λ_1 are functions of Bi
    * $\theta_0 = A_1 e^{-\lambda_1^2 \mathrm{Fo}}$
    * $\theta = \theta_0(\mathrm{Fo}) \cos\left(\lambda_1 \frac{x}{L}\right)$
  - For all the geometries, we can separate θ into a function of time and a function of position

Summary

For non-lumped transient conduction for a plane wall:

$$\theta(x,t) = A_1 e^{-\lambda_1^2 \mathrm{Fo}} \cos\left(\lambda_1 \frac{x}{L}\right)$$

where $\mathrm{Fo} = \dfrac{\alpha t}{L^2}$ is the Fourier number, $\alpha = \dfrac{k}{\rho c}$ is the thermal diffusivity, and A_1, λ_1 are functions of $\mathrm{Bi} = \dfrac{hL}{k}$, the Biot number, which can be determined through a table