Lecture 27, Nov 17, 2022

Fin Sizing

- When can we assume a fin to be infinitely long?
- Compare adiabatic tip vs. infinitely long fin equation, taking the ratio we get $\tanh(aL)$
- For $aL = 1$, $\tanh(aL) = 0.762$; for $aL = 5$, $\tanh(aL) = 0.9999$)
  - As a rule of thumb if $aL \geq 5$ we can assume the fin is infinitely long
  - A value of 1 gives 76.2% of the total heat transfer from an infinitely long fin, with a lot less material
- $L = \frac{1}{a}$ is typically a reasonable length for a fin
- What about the area of fins?
  - Consider area with fin and area without fin
  - $\dot{Q}_{total} = \dot{Q}_{nofin} + \dot{Q}_{fin}$
  - From the fin efficiency definition $\eta_{fin} = \dfrac{\dot{Q}_{fin}}{hA_{fin}(T_b - T_\infty)}$
  - $\dot{Q}_{fin} = h\eta_{fin}A_{fin}(T_b - T_\infty)$
  - Putting it all together $\dot{Q}_{total} = h(A_{nofin} + \eta_{fin}A_{fin})(T_b - T_\infty)$
- $(A_{nofin} + \eta_{fin}A_{fin})$ is the *effective area for heat transfer*
  - In terms of thermal resistances $R = \dfrac{T_b - T_\infty}{\dot{Q}_{total}} = \dfrac{1}{h(A_{nofin} + \eta_{fin}A_{fin})}$

Transient Heat Conduction (Lumped)

- Simple example: taking a material and immersing it in a fluid with a high temperature difference, resulting in rapid heat transfer
  - What is $T(t)$?
- Recall $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$
- Lumped capacitance: simple assumption that there is no temperature gradient in the body, i.e. $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{\partial^2 T}{\partial y^2} = \dfrac{\partial^2 T}{\partial z^2} = 0$, the temperature is at uniform temperature everywhere
  - In reality modes of heat transfer include conduction within the material and convection to the surrounding fluid
  - This makes sense only if $R_{cond} \ll R_{conv}$
- Energy balance: Let $\Delta\dot{E}(t) = -\dot{Q}_{conv}(t)$ be the energy change in the body
  - $\Delta\dot{E}(t) = mc_p\dfrac{\mathrm{d}T}{\mathrm{d}t} = \rho Vc_p\dfrac{\mathrm{d}T}{\mathrm{d}t}$
  - $hA(T - T_\infty) = -\rho Vc_p\dfrac{\mathrm{d}T}{\mathrm{d}t} \implies \dfrac{\mathrm{d}T}{T - T_\infty} = \dfrac{\mathrm{d}(T - T\infty)}{T - T_\infty} = -\dfrac{hA}{\rho Vc_p}\,\mathrm{d}t$
  - Integrate to get $\ln(T - T_\infty) = -\dfrac{hA}{\rho Vc_p}t + c_1$, apply boundary condition that $T(0) = T_i$
  - $\ln\left(\dfrac{T - T_\infty}{T_i - T_\infty}\right) = -\dfrac{hA}{\rho Vc_p}$
  - This compares the "changing driving force" against the "max driving force"
- Define the time constant $\tau = \dfrac{\rho Vc_p}{hA}$ so $\dfrac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{t}{\tau}}$

Equation

For lumped transient heat conduction,

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{t}{\tau}}$$

where $\tau = \dfrac{\rho V c_p}{hA}$

Validity of the Lumped Capacitance Assumption

- This only makes sense if $R_{conv} \gg R_{cond}$
- Take $\dfrac{R_{cond}}{R_{conv}} = \dfrac{\frac{L}{kA}}{\frac{1}{hA}} = \dfrac{hL}{k}$
- $\mathrm{Bi} = \dfrac{hL_c}{k}$ is the *Biot number*, a dimensionless quantity
  - L_c is a characteristic length in the direction of conduction, from the midpoint to the wall
  - $L_c = \dfrac{V}{A}$
  - For a sphere $L_c = \dfrac{r}{3}$, for a cylinder (with length $\gg$ radius) $L_c = \dfrac{r}{2}$
- Consider the steady state analogue; for $\mathrm{Bi} \gg 1$, the temperature drops the sharpest over conduction, so lumped capacitance is not valid; for $\mathrm{Bi} \ll 1$, the temperature drops the sharpest over convection, so lumped capacitance is valid
- Typical cutoff is $\mathrm{Bi} < 0.1$
- Example: putting steel rod at 300°C into furnace at 1200°C with $h = 100\mathrm{W/m^2\,K}, D = 0.1m, k = 51.2\mathrm{W/m\,K}, \rho = 7832\mathrm{kg}m^3, c = 541\mathrm{J/kg\,K}$, how long until the rod temperature reaches 800°C?
  - First, check validity of lumped capacitance assumption: $\mathrm{Bi} = \dfrac{hL_c}{k} = \dfrac{100\mathrm{W/m^2k} \cdot \frac{0.05\mathrm{m}}{2}}{51.2\mathrm{W/m\,K}} = 0.05 < 0.1$ so the assumption is valid
  - Using lumped capacitance $\dfrac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{hA}{\rho V c}t} = e^{-\frac{h}{\rho c}\cdot\frac{2}{r}t}$
  - $\ln \dfrac{800 - 1200}{300 - 1200} = \dfrac{-2 \cdot 100\mathrm{W}m^2.K}{7832\mathrm{kg/m^3} \cdot 541\mathrm{J}kg.K}t$
  - $t = 859\mathrm{s}$