Lecture 26, Nov 15, 2022

More On Finned Surfaces

- Why does longer hair not lead to more heat transfer?
  - k for hair is very low, so longer hair leads to a negligible amount of additional heat transfer
  - The additional hair creates a boundary layer that effectively lowers h leading to less convection
- Consider cylindrical fins, increasing the diameter from d to $2d$ results in $\frac{\dot{Q}'}{\dot{Q}} = 2^{\frac{3}{2}}$, an increase in the total heat transfer
  - For heat flux however we get $\frac{\dot{q}'}{\dot{q}} = 2^{-\frac{1}{2}}$, which is lower
  - When designing a heat sink it might be better to have a larger number of smaller fins
    * Note: Having fins that are too small might break up the flow and change h, leading to worse performance

Finite Length Fins

- Method 1: Consider an adiabatic tip (insulated tip)
  - No heat transfer at the tip means the temperature at the tip must be constant
  - $\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=L} = \left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{x=L} = 0$
  - $\theta(x) = c_1 e^{ax} + c_2 e^{-ax} \implies \frac{\mathrm{d}\theta}{\mathrm{d}x} = c_1 a e^{ax} - c_2 a e^{-ax}$
  - Plug in the initial condition, $0 = c_1 a e^{aL} - c_1 a e^{-aL}$
  - At $x = 0, \theta = \theta_b$, so $\theta_b = c_1 + c_2$
  - Solving yields $\frac{\theta(x)}{\theta_b} = \frac{\cosh(a(L-x))}{\cosh(aL)}$
  - Solving for $\dot{Q}$ using Fourier's law yields $\sqrt{hPkA_c}(T_s - T_\infty)\tanh(aL)$
    * Note $\tanh(L) \to 1$ as $L \to \infty$, so this approaches the infinitely long fin equation as the fin gets longer
- Method 2: Use the "corrected length"
  - Have the convection coming out of the tip be idealized as coming out from the fin side
  - Imagine extending the fin by ΔL such that the additional side area ΔLP equals the size of the tip cross-section, now we can assume the tip is adiabatic
  - Corrected length is $L_c = L + \frac{A_c}{P}$
  - Use this L_c with the adiabatic tip solution for T, θ
  - Note this only works well if $A_c \ll L$ so the extension is minimal

Summary

For a fin with finite length L and an insulated tip, then

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cos(a(L-x))}{\cosh(aL)}$$

where

$$a = \sqrt{\frac{hP}{kA_c}}$$

and the total heat transfer through the fin is

$$\dot{Q} = \sqrt{hPkA_c}(T_s - T_\infty)\tanh(aL)$$

For fins with cross sectional area small relative to the length and non-adiabatic tip, use the corrected length $L_c = L + \dfrac{A_c}{P}$

Fin Efficiency

- The most "efficient" fin would have effectively infinite conductivity, so $T(x) = T_b$ for all x so that the convection along the fin is maximized
- In this case the heat transfer is just convection at uniform temperature, $\dot{Q}_{max} = hA_{fin}(T_b - T_\infty) = hPL(T_b - T_\infty)$ ignoring the fin tip
- Define the fin efficiency as $\eta_{fin} = \dfrac{\dot{Q}}{\dot{Q}_{max}}$
  - This is equal to $\dfrac{\sqrt{hPkA_c}(T_b - T_\infty)}{hPL(T_b - T_\infty)} = \dfrac{1}{aL}$ for an infinitely long fin
  - For an adiabatic tip $\dfrac{\sqrt{hPkA_c}(T_b - T_\infty)\tanh(aL)}{hPL(T_b - T_\infty)} = \dfrac{\tanh(aL)}{aL}$
  - As L increases, η_{fin} approaches 0
- The fin effectiveness is defined as $\varepsilon_{fin} = \dfrac{\dot{Q}_{fin}}{\dot{Q}_0}$ where $\dot{Q}_0$ is the heat transfer without the fin, which would be $hA_c(T_b - T_\infty)$
  - For an infinitely long fin $\varepsilon_{fin} = \sqrt{\dfrac{kP}{hA_c}}$
- To increase the fin effectiveness, maximize k and $\dfrac{P}{A_c}$
- When h goes up, the fin effectiveness goes down; fins are the most effective with low h
  - e.g. if we have a boundary between air and water, it's better to have the fin on the air side since h in air is much lower
- Rule of thumb: Fins are worth it if $\varepsilon \geq 2$