Lecture 24, Nov 3, 2022

Heat Conduction in Cylinders and Spheres

- Consider a cylinder with a z axis and a radial axis r, with pipe length L
  - For a long pipe $\dfrac{\mathrm{d}T}{\mathrm{d}z} \ll \dfrac{\mathrm{d}T}{\mathrm{d}c}$ so we can assume 1D heat conduction
- We want to solve the heat conduction equation to get $T(r)$
- Recall $\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r}\right) = 0$ at steady state, with boundary conditions $T(r_1) = T_1, T(r_2) = T_2$
  - Integrate: $r\dfrac{\mathrm{d}T}{\mathrm{d}r} = c_1 \implies \dfrac{\mathrm{d}T}{\mathrm{d}r} = \dfrac{c_1}{r}$
  - Integrate again: $T(r) = c_1 \ln r + c_2$
  - Apply boundary conditions:
    * $T_1 = c_1 \ln r_1 + c_2, T_2 = c_1 \ln r_2 + c_2$
    * Take the difference: $T_1 - T_2 = c_1 \ln \dfrac{r_1}{r_2}$
    * $c_1 = \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}}$
    * Plug this back in and we get $T_2 = \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} + c_2$ so $c_2 = T_2 - \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln r_2$
    * Plug back in and simplify: $T(r) = \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln(r) - \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln r_2 + T_2$
    * $T(r) = \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln \dfrac{r}{r_2} + T_2$
- Consider heat transfer, which is constant at steady state
  - $\dot{Q}_{cond} = -kA_1 \left.\dfrac{\mathrm{d}T}{\mathrm{d}r}\right|_{r=r_1} = -kA_2 \left.\dfrac{\mathrm{d}T}{\mathrm{d}r}\right|_{r=r_2}$
  - $\dfrac{\mathrm{d}T}{\mathrm{d}r} = \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \dfrac{1}{r}$
  - $\dot{Q}_{cond} = -k(2\pi r_1 L)\left(\dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \dfrac{1}{r_1}\right) = \dfrac{2\pi L k}{\ln \frac{r_2}{r_1}}(T_1 - T_2)$
- Define the thermal resistance of a cylinder as $R = \dfrac{T_1 - T_2}{\dot{Q}} = \dfrac{\ln \frac{r_2}{r_1}}{2\pi L k}$
- For a sphere, we can do a similar derivation and get $R = \dfrac{r_2 - r_1}{4\pi r_1 r_2 k}$

Summary

For 1D heat conduction in a cylinder:

$$T(r) = \frac{T_1 - T_2}{\ln\left(\frac{r_1}{r_2}\right)} \ln\left(\frac{r}{r_2}\right) + T_2$$

which gives a thermal resistance of

$$R_{\text{cylinder}} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi L k}$$

For a sphere this is

$$R_{\text{sphere}} = \frac{r_2 - r_1}{4\pi r_1 r_2 k}$$

For heat convection and radiation, the equations are the same as the Cartesian case, but note areas are no longer constant

Convection in Cylinders and Spheres

- Consider water in a pipe, with water $T_{\infty,1}$ and inner heat transfer coefficient h_1 and surrounding air $T_{\infty,2}$ and outer h_2
- For the total heat transfer we need to consider the convection from the water to the pipe, conduction through the pipe and convection to the outside air
- The main thing to watch out for is that the areas are not constant in cylindrical coordinates
- $R_{total} = \frac{1}{2\pi r_1 L h_1} + \frac{\ln \frac{r_2}{r_1}}{2\pi L k} + \frac{1}{2\pi r_2 L h_2}$