Lecture 21, Oct 27, 2022

One Dimensional Heat Conduction Equation

- Consider heat conduction $x \to x + \Delta x$, surface area at x is A_x; what is the temperature as a function of x?
- $\frac{\mathrm{d}h}{\mathrm{d}t} = mc_p \frac{\partial T}{\partial t} = \rho V c_p \frac{\partial T}{\partial t} = \rho c_p A \mathrm{d}x \frac{\partial T}{\partial t}$
- Energy balance: $\rho c_p A \mathrm{d}x \frac{\partial T}{\partial t} = \dot{Q}_x - \dot{Q}_{x+\Delta x}$
  - In terms of heat flux, $\dot{q}_x A_x - \dot{q}_{x+\Delta x} A_{x+\Delta x}$
  - $\rho c_p \frac{\partial T}{\partial t} = -\frac{1}{A}\left(\frac{\dot{q}_x A_x - \dot{q}_{x+\Delta x} A_{x+\Delta x}}{\Delta x}\right)$
  - Take the limit $\Delta x \to 0$: $\rho c_p \frac{\partial T}{\partial t} = -\frac{1}{A}\frac{\partial(\dot{q}A)}{\partial x}$

Cartesian Coordinates

- Consider Cartesian coordinates, constant area, then we can simplify this as $\rho c_p \frac{\partial T}{\partial t} = -\frac{\partial \dot{q}}{\partial x}$
- Putting this into Fourier's law, $\dot{q} = -k\frac{\mathrm{d}T}{\mathrm{d}x}$, we get $\rho c_p \frac{\partial T}{\partial t} = -\frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x}\right)$
  - With a constant k, we get $\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$
  - Alternatively $\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} = \alpha \frac{\partial^2 T}{\partial x^2}$
- $\alpha = \frac{k}{\rho c_p}$ is the thermal diffusivity, with units of m^2/s
  - Higher k conducts heat well so the gradient is sharper
  - ρc_p stores energy well, so a lot of heat can enter the system without changing the temperature much
  - $\alpha = 1.11 \times 10^{-4}\,\mathrm{m}^2/\mathrm{s}$ for copper, $\alpha = 3.4 \times 10^{-7}\,\mathrm{m}^2/\mathrm{s}$
- For steady state, $\frac{\partial T}{\partial t} = 0$ so $\alpha \frac{\partial^2 T}{\partial x^2} = 0$
  - Integrate this and we get that $\frac{\partial T}{\partial x}$ is a constant

Cylindrical Coordinates

- In the radial direction Fourier's law is $\dot{q} = -k\frac{\partial T}{\partial r}$
- $A = 2\pi r L$
- $\rho c_p \frac{\partial T}{\partial t} = -\frac{1}{2\pi r L}\left(\frac{\partial}{\partial r}\left(2\pi r L\left(-k\frac{\partial T}{\partial r}\right)\right)\right) = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$
- $\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$
- For steady state, this simplifies to $\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0 \implies r\frac{\partial T}{\partial r} = 0$

Spherical Coordinates

- $A = 4\pi r^2$, $\dot{q} = -k\frac{\partial T}{\partial r}$
- Doing the same derivation gets us $\rho c_p \frac{\partial T}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$
- $\frac{\partial T}{\partial t} = \frac{\alpha}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$

Summary

1D heat flow equations:

- Cartesian coordinates:
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$
- Cylindrical coordinates (radial):
$$\frac{\partial T}{\partial t} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)$$
- Spherical coordinates (radial):
$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right)$$

where α is the thermal diffusivity, $\alpha = \frac{k}{\rho c_p}$

In general,
$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^n} \frac{\partial}{\partial r} \left(r^n \frac{\partial T}{\partial r} \right)$$
where $n = 0$ for Cartesian, $n = 1$ for cylindrical and $n = 2$ for spherical