Lecture 1, Sep 8, 2022

Fundamental Concepts

- Energy: the capacity to do work
- Work: a transfer of energy
- Energy is a *fundamental concept*
- Fundamental concepts include:
  - Mass
  - Volume
  - Time
  - Energy
  - Entropy
- Fundamental concepts cannot be measured directly
  - e.g. you can't directly measure mass, instead you have to measure weight (force)
- A system has energy if it can lift a mass
- Heat Q is also a form of energy

The Heat Engine

- A heat engine converts heat into work
- From heat engines came the science of thermodynamics
- A heat engine takes heat Q_H from a high temperature source T_H, and produces work W
  - Heat engines necessarily have to dump heat Q_C into a heat sink at temperature T_C
- To maximize efficiency, we need to maximize w and minimize Q_H
  - The First Law of Thermodynamics says that in a heat engine, $Q_H = Q_C + W$, i.e. energy in equals energy out

Definition

The First Law of Thermodynamics: Energy is conserved

Definition

The thermal efficiency of a heat engine: $\eta_{th} = \dfrac{W}{Q_H}$

Entropy

- Early steam engines had $\eta = 4\%$, engines today only have $\eta = 30\%$
  - What is the maximum efficiency we can achieve?
- The cooling wastes energy, so is it really necessary?
  - Carnot: it's not possible to eliminate the heat sink, so efficiency can never truly be 100%
- To make sense of this, Clausius proposed the new property *entropy* S
- Entropy is similar to energy
- When you add heat to a system, the energy change is $\Delta E = Q$; the entropy change is $\Delta S = \dfrac{Q}{T}$

Definition

Entropy $\Delta S = \dfrac{Q}{T}$, the heat added to a system divided by its temperature

- Consider an object A with temperature $T + \Delta T$, and object B with temperature T
  - When they're brought into contact they will exchange heat, so $\Delta E_A = -Q, \Delta E_B = Q$

  - The entropy change is $\Delta S_A = -\dfrac{Q}{T + \Delta T}, \Delta S_B = \dfrac{Q}{T}$
  - $|\Delta S_A| < |\Delta S_B|$, so the total $\Delta S = \Delta S_A + \Delta S_B > 0$
  - Entropy has been generated, but energy is conserved
- In order to destroy entropy, we'd need the heat to go from the colder object to the hotter object
- Entropy can only be created, never destroyed

Definition

The Second Law of Thermodynamics: The entropy of an isolated system always increases: $\Delta S > 0$

- The second law determines what processes happen spontaneously
- To understand what entropy really means, we need to get down to the molecular level
  - S is related to the distribution of energies, whereas E is the sum of energies

Lecture 2, Sep 12, 2022

Concepts and Definitions

- System: any piece of matter or region of space
  - The system is separated from the surroundings by a boundary
  - The surroundings is everything that's not the system
- Types of systems:
  - Open system/control volume: mass and energy can flow freely in and out of the system
  - Closed system/control mass: the amount of mass in the system is fixed, but energy can still cross the boundary
    - Example: A pipe open on both sides
  - Isolated system: no mass or energy crosses the system boundaries
- Property (state variable): any attribute of a system that can be measured without knowing the history of the system
  - Example: position is a property, but work is not a property since it depends on the path taken
  - For properties, infinitesimal changes are denoted by d, otherwise they are denoted by δ (e.g. $\mathrm{d}x, \delta W$)
  - Temperature is a property, but heat is not
  - Anything that crosses the system boundary is not a property (e.g. even though mass is a boundary, mass introduced to an open system is δm)
- Steady state: all properties of a system do not change with time
  - Note the system may be interacting with the surroundings (e.g. a container that is being filled and drained of water at the same rate)
- Equilibrium: when an isolated system is in steady state
  - All isolated systems tend towards equilibrium, where all properties are constant and uniform
- Quasi-equilibrium process (reversible process): when all the parts of a system are in equilibrium all the time during a process
  - e.g. pushing a piston to compress a gas, doing so infinitely slowly so that the pressure in the gas is always uniform
  - A quasi-equilibrium process does the minimum amount of work (and also gets the most work out of a system)
- Pressure, volume and temperature are related by the ideal gas law $PV = nR_uT$
  - Note P is the absolute pressure, which is equal to the gauge pressure plus atmospheric pressure

Lecture 3, Sep 13, 2022

Energy

- Energy can be macroscopic (changes in velocity or position, e.g. potential energy, kinetic energy) or microscopic (changes in temperature or pressure, e.g. heating up a gas, compressing a gas)
  - We will be talking about the microscopic forms today

Definition

Internal energy U are all microscopic (molecular) forms of energy stored in a system (e.g. "thermal energy", "chemical energy", etc)

- The total energy of a system is $E = KE + PE + U$
  - Kinetic, potential, and internal energy are the only ways energy can be stored in a system
  - E is an extensive property (it depends on the mass of the system)

Definition

Extensive properties depend on the mass of the system (e.g. energy, volume); intensive properties do not depend on the mass of the system (e.g. temperature, pressure)

- For any extensive property, we can define an intensive property by dividing by the mass
  - e.g. the specific volume $v = \frac{V}{m}$, the specific energy $u = \frac{U}{m}$

Ideal Gas Model

- Assumptions of the model:
  - Hard spheres moving randomly
  - All with the same mass m_e
  - Collisions are elastic
  - No long range interactions, i.e. the only time the molecules interact is when they collide
    - Intermolecular forces within most gases are negligible since the molecules are so far apart
  - Molecules are point masses (no rotational KE, etc)
    - Good approximation for monoatomic gases and noble gases
- Consider a box, a cube of side L, and a molecule with velocity c hitting a wall and bouncing off
  - $\Delta p_x = (-m_e c_x) - (m_e c_x) = -2m_e c_x$
  - With the change in momentum we can find the force
  - The distance travelled between successive collisions is $2L$ as the particle bounces off the opposite wall and comes back, so time between collisions is $\frac{2L}{c_x}$
  - The force on the wall is then $F = \frac{\Delta p}{\Delta t} = 2m_e c_x \frac{c_x}{2L} = \frac{m_e c_x^2}{L}$
  - Pressure on the wall would be $\frac{F}{A} = \frac{m_e c_x^2}{L} \cdot \frac{1}{L^2} = \frac{m_e c_x^2}{V}$
  - Summing up all the molecules, we get $\frac{1}{3}\frac{m}{V}c_{rms}^2$, where c_{rms} is the root-mean-square velocity
    - Factor of 3 comes from there being 3 components in 3 principle directions
- Now $PV = \frac{1}{3}mc_{rms}^2$
  - As the velocity of molecules increases, momentum and frequency of impact increases, leading to an increase in pressure
- Note $m = NM$ where M is the molar mass, so $PV = \frac{1}{3}NMc_{rms}^2 = NR_uT$
  - This allows us to relate the temperature to what's happening on the molecular level

  - $\frac{1}{2}Mc_{rms}^2 = \frac{3}{2}R_uT$
  - $\frac{1}{2}\frac{M}{N_A}c_{rms}^2 = \frac{1}{2}m_ec_{rms}^2 = \frac{3}{2}\frac{R_u}{N_A}T$
- $k = \frac{R_u}{N_A}$ is the Boltzmann constant

Important

The average kinetic energy of a molecule in the gas is related to the temperature by $\frac{1}{2}m_ec_{rms}^2 = \frac{3}{2}kT$

- This means $\frac{3}{2}nkT$ where n is the number of molecules of gas is the total kinetic energy of the gas

Important

$U = \frac{3}{2}NR_uT$ is the total kinetic energy of all the molecules, or the internal energy

- In mass units this is $\Delta U = \frac{3}{2}mR(T_2 - T_1)$ where R is the gas constant for the specific gas
- This means that a change in temperature is proportional to a change in internal energy
- Compressing a gas does work on it, increasing U, which leads to an increase in T

Important

For all ideal gases, U is a function of T exclusively, and not on pressure

- For any ideal gas $\Delta U = mc(T_2 - T_1)$ where c is the specific heat
  - $c = \frac{1}{m}\frac{\Delta U}{\Delta T}$
- For monoatomic gases we can expect $c = \frac{3}{2}R$

Lecture 4, Sep 15, 2022

The First Law of Thermodynamics

- For a control mass, energy transfer due to a temperature difference is defined as heat Q, and all other forms of energy transfer are defined as work W
  - $Q + W = \Delta E$

Important

Energy transfer to a system is positive; energy transfer from a system is negative

- As a rate equation: $\dot{Q} + \dot{W} = \frac{\mathrm{d}E}{\mathrm{d}t}$
- Modes of work:
  - Boundary work: force acts on the boundary of the system, e.g. expansion and compression of a gas
    - Assuming a quasi-equilibrium process and no friction, then $W = -PV$
    - $\delta W = -P\,\mathrm{d}V$
- Constant volume processes do no work
- For an isothermal process $W = -mRT\ln\left(\frac{V_2}{V_1}\right)$
  - If the gas is expanding, $V_2 > V_1$ and the work is negative, so it does work on the surroundings

Polytropic Processes

- Assume $PV^n = c$, where n, c are constants
- $n = 1$ is an isothermal process (since $PV = mRT$ is constant, so T is constant)
- $n = 0$ is an isobaric process (since then $PV^0 = P = c$)
- $W_{12} = \int_{V_1}^{V_2} \frac{c}{V^n} \, \mathrm{d}V = -c \left[\frac{V^{1-n}}{1-n} \right]_{V_1}^{V_2} = c \left(\frac{V_2^{1-n} - V_1^{1-n}}{1-n} \right)$
- $W_{12} = \frac{P_2V_2 - P_1V_1}{n-1}$ assuming $n \neq 1$

Lecture 5, Sep 19, 2022

Enthalpy

- Consider a hot water tank
    - To do work with it, we can transfer heat to expand a gas, doing work
        - The amount of work depends on the internal energy
    - We can also use the water's pressure
        - If we open the bottom of the tank, we can get more work than if we open the top of the tank
        - The amount of work depends on the pressure
- The ability to do work by a system depends on its pressure and internal energy
- Consider a bit of water coming out of the tank with volume V, mass m, pressure P, area A, moving by a distance L
    - The force exerted by the system to push the liquid out is $F = PA$, so work done is PLA
    - This is called the *flow work* $W_{flow} = pV$
    - The flow work per unit mass is $w_{flow} = \frac{W_{flow}}{m} = \frac{PV}{m} = Pv$ where v is the specific volume
    - The flow carries $u + pv$ where u is the internal energy per unit mass of the water

Definition

Enthalpy $H = U + PV$, is a measure of the potential to do work

- Enthalpy is an extensive property with units of Joules
- Define the specific enthalpy $h = \frac{H}{m}$, an intensive property
- For an ideal gas u is a function of T only, so in this case $h = u + Pv = u + RT$ is a function of T only
- Consider rate of mass $\dot{m}_1$ entering the system and $\dot{m}_2$ exiting the system
    - Rate of energy entering the system is $\dot{m}_1(u_1 + P_1v_1)$
    - Rate of energy exiting the system is $\dot{m}_2(u_2 + P_2v_2)$
- Consider an isovolumetric process; add heat δQ to a system resulting in $\mathrm{d}U$
    - Since this is a constant volume process $\delta w = 0$, so $\delta Q = \mathrm{d}U$, the heat added is directly added to internal energy
- Consider an isobaric process; add heat δQ
    - The system is allowed to expand, so it does work $\delta W = -P \, \mathrm{d}V$
    - Part of the energy put in becomes work
    - $\delta Q + \delta W = \mathrm{d}U \implies \delta Q = \mathrm{d}U + P \, \mathrm{d}V$
    - Note $H = U + PV \implies \mathrm{d}H = \mathrm{d}U + P \, \mathrm{d}V + V \, \mathrm{d}P = \mathrm{d}U + P \, \mathrm{d}V$ for a constant pressure process
    - For an isobaric process the heat added is equal to the change in enthalpy

Specific Heats

- Add heat Q to a mass m, resulting in ΔT, then the specific heat $c_{avg} = \frac{Q}{m\Delta T} = \frac{q}{\Delta T}$
- The relationship between q and ΔT is not necessarily linear, so $c(T)$ is a function of temperature

Definition

The specific heat $c(t) = \frac{\delta q}{\mathrm{d}T}$

- Consider heating a system at constant volume ($\delta q = \mathrm{d}u$)
  - Define the specific heat at constant volume $c_v = \left(\frac{\partial u}{\partial T}\right)_V$
- Define the specific heat at constant pressure $c_p = \left(\frac{\partial h}{\partial T}\right)_P$
- For an ideal gas, we can write these as total derivatives since h and u are functions of T only
- Note $h = u + RT \implies \frac{\mathrm{d}h}{\mathrm{d}T} = \frac{\mathrm{d}u}{\mathrm{d}T} + R \implies c_p = c_v + R$

Important

For an ideal gas, $c_p = c_v + R$

Definition

The specific heat ratio $\gamma = \frac{c_p}{c_v}$

- Assume c_p, c_v are constants, then $\Delta u = c_v \Delta T, \Delta h = c_p \Delta T$
  - Use T_{avg} to look up values of c_p, c_v
  - If we don't know T_2, we can guess T_2, calculate T_{avg} then iterate guess for T_2
- Liquids and solids are incompressible, so $c_p = c_v = c = \frac{\mathrm{d}u}{\mathrm{d}T}$
  - In this case $\mathrm{d}h = \mathrm{d}u + v\,\mathrm{d}P$
  - $\mathrm{d}u = c\,\mathrm{d}T$
  - $\Delta h = \int_{T_1}^{T_2} c\,\mathrm{d}T + \int_{P_1}^{P_2} v\,\mathrm{d}P$, where c and v are both constants
- For incompressible substances enthalpy can be increased by adding heat or increasing the pressure
- If the process is neither isobaric nor isovolumetric:
  - $\Delta u = c_v(T_2 - T_1)$
  - $\Delta h = c_p(T_2 - T_1)$
  - u and h are properties that are path independent, so even if the process is neither isobaric nor isovolumetric, we can equate it to them

Lecture 6, Sep 20, 2022

Control Volumes/Open Systems

- For these systems we need to take into account the mass flowing in/out of a system and the energy the mass carries
- *Steady flow processes* are devices like pumps, compressors, turbines, etc

Mass Balance

- Consider a generic control volume with an inlet and an outlet
  - Fluid coming in with velocity $\boldsymbol{v}$, with cross section A, entering with length $\mathrm{d}x$, mass δm
    - Note δm is used since mass is crossing the boundary, so it's not a property anymore
    - Length of fluid element inside control volume in time $\mathrm{d}t$ is $\mathrm{d}x = \boldsymbol{v}\,\mathrm{d}t$
    - Volume of fluid element is $A\,\mathrm{d}x = A\boldsymbol{v}\,\mathrm{d}t$

    - Mass of fluid element is $\delta m = \rho A \boldsymbol{v}\,\mathrm{d}t$
  - Mass flow rate $\dot{m} = \frac{\delta m}{\mathrm{d}t} = \rho A \boldsymbol{v}$
- Consider a turbine with the fluid expanding as it goes through
  - Pressure goes down as fluid passes through
  - $\dot{m} = \rho A \boldsymbol{v} = \text{const}$
  - Therefore A must be increased to keep $\boldsymbol{v}$ approximately constant, so the flow is smooth
    - Cross section area must be increased to maintain flow rate as gas loses pressure
  - $\dot{m} = \rho A \boldsymbol{v} = \frac{A\boldsymbol{v}}{v}$
    - Density $\rho = \frac{m}{V}$, which is simply $\frac{1}{v}$ where v is the specific volume
    - For an ideal gas $Pv = RT \implies \frac{1}{v} = \frac{P}{RT}$ so $\dot{m} = \frac{PA\boldsymbol{v}}{RT}$
  - At steady state $\dot{m}_{in} = \dot{m}_{out}$
- First law for a control volume:
  - Assume $\dot{m}$ coming in, with $\boldsymbol{v}_1, h_1$, height z_1, outlet has $\boldsymbol{v}_2, h_2, z_2$
  - The system is heated with $\dot{Q}$, and heat is transferred in with $\dot{W}$
    - The work in this case is usually *shaft work* (more on this later)
  - Energy can be transferred with the flow in different ways:
    - Kinetic energy: $ke = \frac{1}{2}\boldsymbol{v}^2$
    - Potential energy: $pe = gz$
    - Internal energy: u
    - Flow work: Pv
    - The last two is combined into enthalpy $h = u + Pv$
  - Flow energy per unit mass is $h + ke + Pe = h + \frac{\boldsymbol{v}^2}{2} + gz$
  - Energy balance at steady state requires $\dot{E}_{in} = \dot{E}_{out} \implies \dot{Q} + \dot{W} + \dot{m}\left(h_1 + \frac{\boldsymbol{v}_1^2}{2} + gz_1\right) = \dot{m}\left(h_2 + \frac{\boldsymbol{v}_2^2}{2} + gz_2\right)$
  - $\dot{Q} + \dot{W} = \dot{m}\left((h_2 - h_1) + \frac{\boldsymbol{v}_2^2 - \boldsymbol{v}_1^2}{2} + g(z_2 - z_1)\right)$
    - Define $q = \frac{\dot{Q}}{\dot{m}}, w = \frac{\dot{W}}{\dot{m}}$
    - $q + w = (h_2 - h_1) + \frac{\boldsymbol{v}_2^2 - \boldsymbol{v}_1^2}{2} + g(z_2 - z_1)$

Definition

The first law for a control volume:

$$\dot{Q} + \dot{W} = \dot{m}\left((h_2 - h_1) + \frac{\boldsymbol{v}_2^2 - \boldsymbol{v}_1^2}{2} + g(z_2 - z_1)\right)$$

or mass normalized:

$$\dot{q} + \dot{w} = (h_2 - h_1) + \frac{\boldsymbol{v}_2^2 - \boldsymbol{v}_1^2}{2} + g(z_2 - z_1)$$

Steady Flow Devices

- A turbine thermodynamically is h_1 going in, h_2 coming out, $\dot{W}_{shaft}$ extracted
- A compressor thermodynamically is just a turbine running backwards
- Consider a turbine with $\dot{Q} = 0$, i.e. heat loss is negligible, and changes in KE and PE are negligible:

  - $\dot{Q} + \dot{W} = \dot{m}\left((h_2 - h_1) + \frac{\boldsymbol{v}_2^2 - \boldsymbol{v}_1^2}{2} + g(z_2 - z_1)\right) \iff \dot{W} = \dot{m}(h_2 - h_1)$
  - For a turbine, T and P at the inlet are high, so $h_1 > h_2$, and $\dot{W} < 0$ (i.e. work is done on the surroundings)
  - Reverse this for a compressor ($h_2 > h_1$), and work is done by the surroundings to the system
- For a turbine/compressor, the work is in the difference of enthalpies
- Example: A gas turbine is supplied 10kg/s of air at $800°\text{C}, 600\text{kPa}$, which leaves at $300°\text{C}, 100\text{kPa}$; what is the power output?
  - Assume ideal gas, then $\Delta h = c_p \Delta T$
  - $\dot{W} = \dot{m} c_p (T_2 - T_1)$
  - $T_{avg} \approx 800\text{K}$, look up c_p
  - $\dot{w} = 10\text{kg/s} \cdot 1.099\text{kJ/kg}°\text{C} \cdot (300 - 800) = -5495\text{kW}$
- For a pump, the goal is usually to change the PE
  - Consider a height change Δz between the inlet and outlet
  - P_1 coming in, P_2 coming out, $\dot{W}$ driving the pump
  - Assume change in KE is negligible, $\dot{Q}$ negligible
    - * For an incompressible fluid, density is constant, so as long as the inlet and outlet pipe diameters are the same, KE is the same
  - $\dot{W} = \dot{m}\left((h_2 - h_1) + g(z_2 - z_1)\right)$
  - For an incompressible fluid, then $h_2 - h_1 = c(T_2 - T_1) + v(P_2 - P_1)$ (we can change enthalpy by change in temperature or pressure)
  - Assume no change in temperature so $h_2 - h_1 = v(P_2 - P_1) \implies \dot{W} = \dot{m}(v(P_2 - P_1) + g(z_2 - z_1))$ (i.e. frictional losses are negligible)
    - * The work put in is taken out as either a change in pressure or a change in PE
    - * If the difference in height between the inlet/outlet is small, then the pressure change is large
- Nozzles and diffusers change the KE (nozzles increase KE by having the inlet larger than the outlet, diffusers decrease KE by having a nozzle backwards)
  - Assume PE change is 0, no heat or work going in
  - $0 = \dot{m}\left((h_2 - h_1) + \frac{\boldsymbol{v}_2^2 - \boldsymbol{v}_1^2}{2}\right)$
  - Assume $\boldsymbol{v}_2 \gg \boldsymbol{v}_1$, so we can neglect $\boldsymbol{v}_1^2$
  - $\boldsymbol{v}_2 = \sqrt{2(h_1 - h_2)}$
  - Enthalpy is converted into KE by increasing velocity
    - * Pressure decreases

Lecture 7, Sep 22, 2022

Converting Heat into Work

- Every gas has internal energy, so can we use the air to do work?
- Can we take Q from the air and convert this into W?
  - This does satisfy energy conservation
  - Of course, this is not possible, so why?
- A heat engine involves a heat source T_H, transferring heat Q_H to a gas to expand it, pushing a piston up
  - The engine stops when the temperature of the gas T reaches T_H
  - If we want the engine to do work again, we have to cool the gas, which requires a heat sink T_C
- As a consequence:
  1. To increase work, increase T_H
  2. In the limit $T_H \to T_C$, no work is done
  3. We always need 2 temperatures
- This implies that there is a "quality" of energy – just because something has internal energy, doesn't mean we can actually do useful work with it

  - The higher the temperature, the more work we can get out of it
- Can we define a property that measure this?
  - This would require some combination of the amount of energy and the temperature

Entropy and the Second Law

Definition

The fundamental property entropy S, where $\mathrm{d}S = \dfrac{\delta Q}{T}$, with units of J/K

- Entropy is an extensive property that changes when heat is added or remove from a system
- The entropy change is the ratio of the heat added and the temperature at which it was added
  - If δQ is added to a system, then T is the temperature of the boundary where the heat crossed
  - An increase in temperature decreases the entropy change
- Assume no work done so $\mathrm{d}U = \delta Q$, then the entropy change $\mathrm{d}S = \dfrac{\delta Q}{T}$
- $\Delta S = \displaystyle\int_1^2 \mathrm{d}S = \int_1^2 \frac{\delta Q}{T}$

Definition

A thermal reservoir is a system whose temperature remains constant, even when heat is added or taken from it (think of a very large thermal mass, e.g. the air, a lake, etc)

- For a thermal reservoir, $\Delta S = \displaystyle\int_1^2 \frac{\delta Q}{T} = \frac{1}{T}\int_1^2 \delta Q = \frac{Q_{12}}{T}$
- Consider two thermal reservoirs A with $T + \Delta T$ and B with T (assume $\Delta T > 0$); bring them together and let them exchange heat
  - $\Delta S_A = \dfrac{Q}{T + \Delta T}$
  - $\Delta S_B = \dfrac{Q}{T}$
  - ΔS_B is greater in magnitude, so the total entropy increases (entropy generated is $S_{gen} = |\Delta S_B| - |\Delta S_A|$)
  - Note $\Delta T \to 0, \implies S_{gen} \to 0$
- Energy is conserved, but entropy can be generated
  - Entropy is generated when heat transfer occurs between two thermal reservoirs whose temperatures differ by a finite amount
  - All real heat transfer processes generate entropy
- Entropy cannot be destroyed, as that would involve a heat transfer from a lower temperature to a higher temperature
- Consider an isolated system not in equilibrium (system contains cold gas, and a little pocket of hot gas)
  - When the system goes to equilibrium, the heat from the hot pocket goes to the rest of the gas, so the entropy increases
- Doing work on a system does not change its entropy

Definition

The Second Law of Thermodynamics: The entropy an isolated system will increase until the system reaches equilibrium; the entropy of an isolated system remains constant

$\mathrm{d}S_{isolated} > 0$ when not in equilibrium; $\mathrm{d}S_{isolated} = 0$ when in equilibrium; $\mathrm{d}S_{isolated} < 0$ is impossible

- The second law defines what things happen spontaneously

- Example: a ball can be dropped and have its PE be lost to heat, but a ball on the ground can never spontaneously gain PE from the environment and lift up
    - If the ball spontaneously jumps into the air, heat has to be taken from the air, which decreases the entropy, violating the second law

Real Processes

- Real processes always produce entropy
- Consider a force pulling up a mass $F = mg$ at equilibrium
    - If we increase the force to $F + \Delta F$ in the limit as $\Delta F \to 0$, this would be a quasi equilibrium process and the mass is lifted up
    - If we reverse this process and make ΔF negative, then we can recover all the energy
    - For this reason the process is called *reversible*

Definition

Reversible process: Both the system and its surroundings can be put back into the initial state

- In a real system, heat is generated by friction in the pulley
    - Q is transferred to the surroundings, so the entropy of the surroundings increases
    - Even if the mass is lowered back, the entropy of the surroundings cannot be lowered again

Definition

Isentropic process: Quasi-equilibrium processes (reversible) without heat transfer or generation of entropy

Lecture 8, Sep 26, 2022

Molecular Definition of Entropy

- Consider a rigid, insulated box divided into two parts: one with gas, the other with vacuum
    - When the partition is removed, the gas will expand and the pressure equalizes
    - This is accompanied by an increase in entropy
    - But why do the molecules spread out evenly?
- A *microstate* is a combination of the exact states of molecules (we cannot directly measure this); a *macrostate* is some kind of an average (we can measure this in terms of e.g. pressure)
- Divide the boxes into left and right, and consider if there were just 2 particles in the left box
    - We can either have both on one side and one on each side
    - Consider all the possibilities (*microstates*): LL, LR, RL and RR; in terms of *macrostates* (averages), the macrostate with an even distribution is more likely than all the molecules to one side
- If there are instead 4 particles, then there are 5 macrostates (4 left, 3 left + 1 right, etc)
    - The probability of all the particles being on one side is even smaller
- A probability distribution of the percentage of particles on one side would follow a Gaussian
    - The probability of having everything on one side decreases as the number of particles increases
    - The total number of states is 2^n, but there is only one state corresponding to all particles on one side
    - For a large number of particles the distribution is extremely sharp and essentially all cases are at equilibrium
- The equilibrium macrostate is the one that has the largest number of microstates (and thus the highest probability)
    - This leads us to define molecular entropy in terms of the number of microstates that correspond to each macrostate, so that as the system moves toward equilibrium, it increases
- For a very large number of particles, $S_{eq} = \Omega_{total}$

- But note entropy is an extensive property, i.e. if we have two systems with entropies S_1 and S_2, the entropy of the combined system should be $S_1 + S_2$
    - Consider if each system had N particles, then the combined system has $2N$ particles, so the number of microstates would be 2^{2N}
    - This prompts us to use logarithms

Equilibrium in a Gas

- Consider a gas in an isolated system
    - Constant mass defined by n, number of molecules
    - Constant energy determined by U, the internal energy
        - Note the energies of the molecules are not continuous but occupy discrete levels
- $n = \sum_i n_i$ where n_i is the number of molecules with energy ε_i
    - The total energy is $U = \sum_i \varepsilon_i n_i$
- Any arrangement of molecules that satisfies these constraints is a microstate
- At equilibrium the molecules will be in a *Maxwell-Boltzmann distribution*

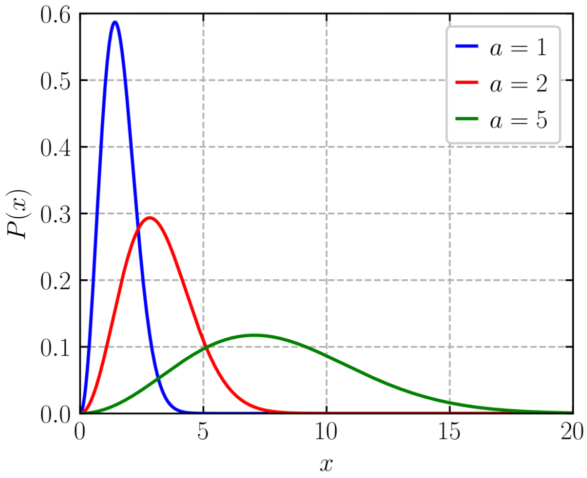

Figure 1: Maxwell-Boltzmann distribution plotted as a function of energy

- As the temperature goes up, the peak of the distribution moves to the right; more energy levels become available, and the distribution flattens out
- If Ω is the number of microstates that correspond to equilibrium state

Definition

$S = k \ln \Omega$, where Ω is the number of microstates of the system, and k is the Boltzmann constant, $k = 1.38 \times 10^{-23}$ J/K

- The Boltzmann constant makes this definition consistent with the Clausius definition based on heat transfer
- Consider the example at the beginning; we had Ω_i microstates before the partition was opened and Ω_f after the partition was opened ($\Omega f \gg \Omega i$)
    - This represents a positive entropy change of $\Delta S = k \ln \left(\frac{\Omega_f}{\Omega_i} \right)$
- To calculate Ω, we need to solve Schrödinger's Equation
- For a monoatomic ideal gas, with n molecules, volume of V and energy U, then $\Omega(U, V, n) = f(n) V^n U^{\frac{3}{2}n}$

  - Thus $S(U, V, n) = nk \ln V + \frac{3}{2} nk \ln U + k \ln f(n)$
  - $\Delta S = nk \left(\ln \frac{V_2}{V_1} + \frac{3}{2} \ln \frac{U_2}{U_1} \right)$
  - Recall $nk = NR_u = mR$ and $U \propto T$, therefore $\Delta S = mR \left(\ln \frac{V_2}{V_1} + \frac{3}{2} \ln \frac{T_2}{T_1} \right)$

Important

For a monoatomic ideal gas, $\Delta S = mR \left(\ln \frac{V_2}{V_1} + \frac{3}{2} \ln \frac{T_2}{T_1} \right)$

Lecture 9, Sep 27, 2022

More on Entropy

- Suppose we have a pure substance at absolute zero; all the molecules go to their lowest energy state
  - This means we only have 1 microstate, so $S = k \ln \Omega = 0$
- Entropy, unlike energy, has an absolute zero value
- At the molecular level this can be derived; but on a macroscopic level it needs to be a postulate

Definition

The Third Law of Thermodynamics: The entropy of a pure substance in thermodynamic equilibrium is zero at a temperature of 0K

The State Postulate

- How much information is contained in the state of a system? Is there a minimum list of variables from which you can calculate everything else?
  - We need mass of the system and its energy
  - We need one property for each mode of work (e.g. boundary work is specified by the volume; spring work is specified by the spring compression, etc)
    - *Simple compressible systems* are systems where the only mode of work is boundary work; for these we only need the volume
  - We need heat transfer, which can be specified by the internal energy U (since we already have work)

Definition

The State Postulate: The equilibrium state of a pure, simple compressible system is completely described by its: mass m, volume V and internal energy U

- Since entropy is a property this means $S = S(U, V, m)$
- Define the *specific entropy* $s = \frac{S}{m}$, which is a function of volume and energy: $s = s(u, v)$
  - Or $u = u(s, v)$ or $v = v(u, s)$
- In general to fix the state of a pure, simple compressible system, we need two independent intensive properties
- These 4 postulates are all we need to develop thermodynamics

Entropy Changes With Heat and Work

- Transfer of heat creates entropy, but work does not, why?

- Consider the energy levels ε_i each with n_i molecules: $U = \sum_i n_i\varepsilon_i \implies \mathrm{d}U = \sum_i \varepsilon_i \mathrm{d}n_i + \sum_i n_i \mathrm{d}\varepsilon_i$
  - $\mathrm{d}n_i$ is a change in the number of molecules in each level – the rearrangement of molecules in energy levels
    - * This would correspond to heating up the gas to shift the peak of the Maxwell-Boltzmann distribution to the right
    - * The shifting of the distribution means now there are new energy states occupied, increasing the number of microstates
  - $\mathrm{d}\varepsilon_i$ is a change of the energy of the energy levels
    - * This would correspond to lifting up the gas so every energy state now has more potential energy (the entire distribution gets shifted evenly)
    - * This does not occupy new energy states, so the number of microstates stays the same
  - $\delta W = \sum_i n_i \mathrm{d}\varepsilon_i, \delta Q = \sum_i \varepsilon_i \mathrm{d}n_i \implies \mathrm{d}U + \delta W + \delta Q$

Reversible and Irreversible Processes

- A process that produces no entropy is reversible; conversely a process that process entropy is irreversible
  - e.g. in a rapid compression the molecules near the piston are compressed more than the others, which increases their energy by a greater amount; some molecules are shifted into higher energy states and entropy increases
  - On the other hand a quasi-static process has all the molecules compressed the same amount, moving the energies up but not redistributing them, so no entropy is created
- In reality all real processes are irreversible and generate entropy
- The universe is an isolated system, so $\Delta S > 0$
  - Initially all mass and energy is concentrated in a single point, which has very low entropy; after the big bang energy and mass are dispersing and energy increases
  - The universe tends to equilibrium, where all mass and energy are evenly distributed; at this point there is no more energy and mass gradient, so no work can be done and no processes happen

“Order” and “Disorder”

- When we say something is “orderly” and “disorderly”, how do we actually define this?
  - e.g. a deck of hards has 52! of shuffling, and every combination has a probability of 1/52! but why do we think some are more “orderly” than others?
- We would need to define our microstates and macrostates
  - Microstates in this case is every combination of the cards
  - Microstates need to be changing randomly
  - The macrostate can be any macroscopic property, e.g. the number of black cards in the top half of the deck divided by 26
- Correct statement: *Isolated* systems that are *randomly alternating between microstates* go spontaneously from a macrostate that corresponds to a smaller number of microstates to a macrostate with a larger number of microstates
  - It’s incorrect to say that systems simply go from being highly organized to being disorganized

Summary

The Postulates of Classical Thermodynamics

The Fundamental Concepts:

- Mass m
- Volume V
- Energy U
- Entropy S

The Four Postulates:

- First Law: $Q + W + \Delta E$
- Second Law: $\Delta S \geq 0$ for an isolated system
- Third Law: $S = 0$ for a pure substance at absolute zero
- State Postulate: $S = S(U, V, m)$

From these fundamental concepts and postulates, we can define everything else.

Lecture 10 (Videos)

Postulate of Thermodynamics

- *Fundamental properties* are properties that cannot be explained in simpler terms:
  - By defining them we are postulating fundamental assumptions
- Fundamental properties (postulates):
  - Mass: matter exists
  - Volume: matter takes up space
  - Energy: matter moves and interacts
  - Entropy: interactions between pieces of matter lead to equilibrium
- Other properties:
  - Time is a fundamental property, but we don't care about it since we only deal with equilibrium states which are independent of time
  - All other properties can be expressed in terms of fundamental properties (including temperature and pressure)
- Four Postulates:
  - State Postulate: All isolated systems reach a state of equilibrium; the equilibrium state of a pure, simple compressible substance is completely described by its mass, volume, and internal energy
    - $S = S(U, V, m)$
  - First Law: The change in energy of a closed system equals the net energy transferred to it in the form of work and heat
    - $Q + W = \Delta E$
  - Second Law: The entropy of an isolated system increases until equilibrium, at which point it remains constant
    - $\Delta S \geq 0$
  - Third Law: The entropy of a pure substance in equilibrium at absolute zero is zero
    - $S = 0$ at $T = 0$
- Using these fundamental properties and postulates we can develop thermodynamics as a logical system

Defining Temperature

- T can be defined in terms of fundamental properties
- Consider system A and B at different temperatures, brought together to form system C; what is the condition for C to reach thermal equilibrium?
  - The two systems have m_A, V_A, m_B, V_B fixed, so only U_A, U_B may change
  - $S_C = S_A(U_A, V_A, m_A) + S_B(U_B, V_B, m_B) \implies \mathrm{d}S_C = \dfrac{\partial S_A}{\partial U_A}\mathrm{d}U_A + \dfrac{\partial S_B}{\partial U_B}\mathrm{d}U_B$

  - At equilibrium $\mathrm{d}S_C = 0$ (entropy is constant)
  - C is isolated so $U_C = U_A + U_B = \text{const} \implies \mathrm{d}U_A = -\mathrm{d}U_B$
  - Substituting this we have $\mathrm{d}S_C = \left(\frac{\partial S_A}{\partial U_A} - \frac{\partial S_B}{\partial U_B}\right)\mathrm{d}U_A = 0$
  - Therefore for thermal equilibrium we must have $\frac{\partial S_A}{\partial U_A} = \frac{\partial S_B}{\partial U_B}$ – what does this mean?

Definition

Temperature $T = \left(\frac{\partial U}{\partial S}\right)_{m,V}$

- The equilibrium condition becomes $\frac{\partial S_A}{\partial U_A} = \frac{\partial S_B}{\partial U_B} \implies \frac{1}{T_A} = \frac{1}{T_B} \implies T_A = T_B$
  - This satisfies our understanding of temperature since the system is at thermal equilibrium when temperatures are the same
- If $T_A > T_B$ then $\mathrm{d}S_C = \left(\frac{1}{T_A} - \frac{1}{T_B}\right)\mathrm{d}U_A > 0$ must be true; to satisfy this $\mathrm{d}U_A < 0$
  - This means energy of system A decreases while energy of B increases
  - This matches our understanding since we have heat transfer from A to B

Defining Pressure

- Consider a container with two compartments A and B separated by a locked piston, initially with different pressures; when the piston is removed the system goes to equilibrium; what is the equilibrium condition?
  - $S_C = S_A(U_A, V_A, m_A) + S_B(U_B, V_B, m_B) \implies \mathrm{d}S_C = \frac{\partial S_A}{\partial U_A}\mathrm{d}U_A + \frac{\partial S_A}{\partial V_A}\mathrm{d}V_A + \frac{\partial S_B}{\partial U_B}\mathrm{d}U_B + \frac{\partial S_B}{\partial V_B}\mathrm{d}V_B$
  - Energy and volume are constant so $\mathrm{d}U_A = -\mathrm{d}U_B, \mathrm{d}V_A = -\mathrm{d}V_B$
  - $\mathrm{d}S_C U = \left(\frac{\partial S_A}{\partial U_A} - \frac{\partial S_B}{\partial U_B}\right)\mathrm{d}U_A + \left(\frac{\partial S_A}{\partial V_A} - \frac{\partial S_B}{\partial V_B}\right)\mathrm{d}V_A = \left(\frac{1}{T_A} - \frac{\partial 1}{\partial T_B}\right)\mathrm{d}U_A + \left(\frac{\partial S_A}{\partial V_A} - \frac{\partial S_B}{\partial V_B}\right)\mathrm{d}V_A$
  - For a system at equilibrium $T_A = T_B$ so $\frac{\partial S_A}{\partial V_A} = \frac{\partial S_B}{\partial V_B}$

Definition

The thermodynamic pressure P, such that $\frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_{m,U} \implies P = T\left(\frac{\partial S}{\partial V}\right)_{m,U}$

- Substituting this back in we have $\mathrm{d}S_C = \left(\frac{1}{T_A} - \frac{1}{T_B}\right)\mathrm{d}U_A + \left(\frac{P_A}{T_A} - \frac{P_B}{T_B}\right)\mathrm{d}V_A$
  - At equilibrium we would require $T_A = T_B, P_A = P_B$
  - If the system is not at equilibrium (assume $T_A = T_B = T$, but $P_A > P_B$), then $\mathrm{d}S_C = \left(\frac{P_A}{T} - \frac{P_B}{T}\right)\mathrm{d}V_A > 0 \implies (P_A - P_B)\mathrm{d}V_A > 0 \implies \mathrm{d}V_A > 0$
  - This means system A, at higher pressure, expands while system B contracts, which matches our understanding of pressure

Calculating Entropy Changes

- In terms of intensive properties $\frac{1}{T} = \left(\frac{\partial s}{\partial u}\right)_v, \frac{P}{T} = \left(\frac{\partial s}{\partial v}\right)_u$
- $s = s(u, v) \implies \mathrm{d}s = \left(\frac{\partial s}{\partial u}\right)_v \mathrm{d}u + \frac{\partial s}{\partial v}u\mathrm{d}v = \frac{1}{T}\mathrm{d}u + \frac{P}{T}\mathrm{d}v$

Important

The Gibbs Equation: $\mathrm{d}s = \frac{1}{T}\mathrm{d}u + \frac{P}{T}\mathrm{d}v$

Alternative form: $\mathrm{d}s = \frac{\mathrm{d}h}{T} - \frac{v}{T}\mathrm{d}P$

- This gives us the change in entropy as a function of things we can easily measure
- For an impressible substance, $\mathrm{d}v = 0 \implies \mathrm{d}s = \frac{1}{T}\mathrm{d}u$ and $c_p = c_p = c$ and $\mathrm{d}u = c(T)\mathrm{d}T$, so $\mathrm{d}s = c(T)\frac{\mathrm{d}T}{T}$
  - $\Delta s = s_2 - s_1 = \int_{T_1}^{T_2} c(T)\frac{\mathrm{d}T}{T}$
  - $\Delta s = c_{avg} \ln \frac{T_2}{T_1}$
  - Notice $\Delta s = c_{avg}(\ln T_2 - \ln T_1)$, compare to $\Delta u = c_{avg}(T_2 - T_1)$
- For an ideal gas $\mathrm{d}u = c_v(T)\mathrm{d}T$ and $\frac{P}{T} = \frac{R}{v} \implies \mathrm{d}s = \frac{c_v(T)}{T}\mathrm{d}T + R\frac{\mathrm{d}v}{v}$
  - $\Delta s = \int_{T_1}^{T_2} c_v(T)\frac{\mathrm{d}T}{T} + \int_{v_1}^{v_2} R\frac{dv}{v}$
  - Assume constant c_v, then $\Delta s = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1}$
  - This gives $\Delta s = \Delta s(T, v)$
  - From the ideal gas equation $\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{v_2}{v_1}$ so $\Delta s = c_v \ln \frac{P_2}{P_1} + (c_v + R) \ln \frac{V_2}{V_1}$
  - This gives $\Delta s = \Delta s(P, v) = c_v \ln \frac{P_2}{P_1} + c_p \ln \frac{v_2}{v_1}$
  - For $\Delta s(P, T)$: $T\mathrm{d}s = \mathrm{d}u + P\mathrm{d}v$, $h = u + Pv \implies \mathrm{d}h = \mathrm{d}u + P\mathrm{d}v + v\mathrm{d}P \implies \mathrm{d}u + P\mathrm{d}v = \mathrm{d}h - v\mathrm{d}P \implies T\mathrm{d}s = \mathrm{d}h - v\mathrm{d}P$
    - Alternative form of the Gibbs equation: $\mathrm{d}s + \frac{\mathrm{d}h}{T} - \frac{v}{T}\mathrm{d}P$
  - For an ideal gas $\mathrm{d}h = c_p(T)\mathrm{d}T$ and $\frac{v}{T} = \frac{R}{P}$ so $\mathrm{d}s = \frac{c_p(T)}{T}\mathrm{d}T - R\frac{\mathrm{d}P}{P}$
    - $\Delta s(P, T) = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$

Summary

For constant specific heat, for an ideal gas:

- $\Delta s(T, v) = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1}$
- $\Delta s(P, v) = c_v \ln \frac{P_2}{P_1} + c_p \ln \frac{v_2}{v_1}$
- $\Delta s(P, T) = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$

For an incompressible substance, $\Delta s = c_{avg} \ln \frac{T_2}{T_1}$

- For variable specific heat, consider $\int_{T_1}^{T_2} c_p(T)\frac{\mathrm{d}T}{T} = \int_{T_{ref}}^{T_2} c_p(T)\frac{\mathrm{d}T}{T} - \int_{T_{ref}}^{T_1} c_p(T)\frac{\mathrm{d}T}{T}$

- Define $s^0(T_0) = \int_{T_{ref}}^{T_0} c_p(T)\frac{\mathrm{d}T}{T}$, so that we can instead use $s^0(T_2) - s^0(T_1)$ instead of $c_v \ln\frac{T_2}{T_1}$ whenever specific heat is nonconstant
  - Air tables list these values of s^0
  - Usually $s^0 = 0$ at $T_{ref} = 0\text{K}$

Important

When c_p is nonconstant, instead of $c_p \ln\frac{T_2}{T_1}$, use $s^0(T_2) - s^0(T_1)$

Lecture 11, Oct 3, 2022

Isentropic Processes

- For incompressible substances $\Delta s = c_{avg}\ln\frac{T_2}{T_1}$, then for an isentropic process $\Delta s = 0 \implies T_2 = T_1$
- Internal irreversibility: something within the system converting work into heat
  - An isentropic system is internally reversible
  - In some process, the less entropy you generate, the better it is (the ideal case would be completely isentropic)
- For an ideal gas $\Delta s = 0 \implies c_v \ln\frac{T_2}{T_1} = -R\ln\frac{v_2}{v_1} \implies \frac{T_2}{T_1} = \left(\frac{v_1}{v_2}\right)^{\frac{R}{c_v}}$
  - Recall $\gamma = \frac{c_p}{c_v}$ and $c_p - c_v = R$ so $\frac{T_2}{T_1} = \left(\frac{v_1}{v_2}\right)^{\gamma-1}$
  - We can also show that $\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$
  - Combining the two we get $\frac{P_2}{P_1} = \left(\frac{v_1}{v_2}\right)^{\gamma} \implies P_1v_1^\gamma = P_2v_2^\gamma = \text{const}$
  - This is the equation for a polytropic process!

Important

A polytropic process ($Pv^n = \text{const}$) is isentropic if $n = \gamma$

Summary

For an isentropic process:

- $\frac{T_2}{T_1} = \left(\frac{v_1}{v_2}\right)^{\gamma-1}$
- $\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$
- $\frac{P_2}{P_1} = \left(\frac{v_1}{v_2}\right)^{\gamma}$

Where $\gamma = \frac{c_p}{c_v}$

Entropy Balance in a Control Mass

- $\Delta S = S_{in} - S_{out} + S_{gen}$, at equilibrium this is equal to 0
  - Entropy can be transferred in by heating the system and transferred out by cooling it down

- If the system is internally reversible, then $\Delta S = S_{in} - S_{out}$
- On a rate basis $\frac{\mathrm{d}S}{\mathrm{d}t} = \dot{S}_{in} - \dot{S}_{out} + \dot{S}_{gen}$
    - For a control mass the only way entropy can be transferred is via heat, so $\dot{S}_{heat} = \frac{\dot{Q}}{T}$ (T is the temperature of the boundary where heat crosses)
    - $\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_j \frac{\dot{Q}_j}{T_j} + \dot{S}_{gen}$ where T_j are the local temperatures of the boundaries at which the heat crosses

Lecture 12, Oct 4, 2022

More on Entropy Balance

- Recall $\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_j \frac{\dot{Q}_j}{T_j} + \dot{S}_{gen}$
- Consider a control volume where $\dot{m}_i, \dot{m}_e$ are the mass entry and exit rates
    - In addition to the heat transfer, the internal irreversibilities generating entropy, there is also entropy carried in by the mass
    - Entropy balance is then $\frac{\mathrm{d}s}{\mathrm{d}t} = \sum_j \frac{\dot{Q}_j}{T_j} + \sum_i \dot{m}_i s_i - \sum_e \dot{m}_e s_e + \dot{s}_{gen}$
        - * Reversible system means $\dot{s}_{gen} = 0$
        - * Adiabatic process has $\dot{Q} = 0$
        - * Steady state systems have $\frac{\mathrm{d}s}{\mathrm{d}t} = 0$
        - * With single-entry and single-exit systems $\dot{m}_i = \dot{m}_e$

Summary

Entropy balance for a control mass: $\frac{\mathrm{d}s}{\mathrm{d}t} = \sum_j \frac{\dot{Q}_j}{T_j} + \dot{s}_{gen}$

For a control volume: $\frac{\mathrm{d}s}{\mathrm{d}t} = \sum_j \frac{\dot{Q}_j}{T_j} + \sum_i \dot{m}_i s_i - \sum_e \dot{m}_e s_e + \dot{s}_{gen}$

Isentropic Steady Flow Devices

- Consider a burner outputting $\dot{m}, P_1, T_1$, how much work can we get out of it if we connect this to a turbine?
    - We know the inlet conditions, but we can't simply do $\dot{m}c_p(T_2 - T_1)$ because we don't have T_2
    - We know $P_2 = P_{atm}$
    - In the most ideal case, if we consider the turbine to have no internal irreversibilities and completely adiabatic, then the process is isentropic and $s_1 = s_2$
    - This allows us to get the work in the most ideal case

Isentropic Efficiency

- In real life nothing is really isentropic, so how do we relate the numbers for the most ideal case to real life?
- Recall $s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$

    - Assume constant pressure then $T_2 = T_1 \exp\left(\frac{s_2 - s_1}{c_p}\right)$
- The curve of T with respect to s depends on P; at P_1 and P_2 there are two separate curves
    - By dropping down from T_1, s_1 vertically (isentropic) we can find T_2
    - In reality (non-isentropic process) we'd have to move to the right since entropy increases, which gives us a different (higher) T_2

Definition

Isentropic efficiency for a turbine: $\eta_t = \frac{\left(\frac{\dot{W}}{\dot{m}}\right)_{\text{actual}}}{\left(\frac{\dot{W}}{\dot{m}}\right)_{\text{ideal}}}$

For a compressor this would be $\eta_c = \frac{\left(\frac{\dot{W}}{\dot{m}}\right)_{\text{ideal}}}{\left(\frac{\dot{W}}{\dot{m}}\right)_{\text{actual}}}$

- Assume constant c_p then $\eta_t = \frac{h_2 - h_1}{h_{2s} - h_1} = \frac{c_p(T_2 - T_1)}{c_p(T_{2s} - T_1)} = \frac{T_2 - T_1}{T_{2s} - T_1}$
    - This is easy to measure and is typically provided by a turbine manufacturer (typically 80% to 90%)
- Do this in reverse and it works for a compressor
- For a nozzle it's defined in terms of KE: $\eta_{nozzle} = \frac{\frac{V_2^2}{2}}{\frac{V_{2s}^2}{2}}$

Bernoulli's Equation

- Assume *inviscid* flow (no viscosity, no friction), so it can be modelled as a reversible flow
- Consider fluid entering a pipe at V_1, P_1, z_1, T_1 and exiting at V_2, P_2, z_2, T_2
- Assume flow is isentropic, incompressible
    - This is generally true for gases that aren't moving too fast
- This means $\Delta s = c \ln \frac{T_2}{T_1} = 0 \implies T_2 = T_1$
- $h_2 - h_1 = c(T_2 - T_1) + v(P_2 - P_1) = v(P_2 - P_1)$
- The energy balance equation becomes $\frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) = 0$
    - Rearrange to $\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$
    - This means this is a constant

Definition

Bernoulli's Equation: $\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{const}$ (where V is the velocity)

- If we assume z is roughly constant, then as velocity goes up, pressure goes down

Lecture 13, Oct 6, 2022

Phase Change

- We assume a phase is either liquid, vapour, or solid and only deal with pure substances
- Most applications involve phase changes

  - e.g. cooling down a hot surface by blowing a fluid across the surface (surface cooling)
    - If we used air, $c_p \approx 1\text{kJ/kg K}$, but if we used water and get it to boil, the latent heat is approximately 2000kJ/kg
    - If we can get the water to boil we can carry away orders of magnitude more heat
  - Steam turbines are also an example

Two-Phase Mixtures

- e.g. getting vapour bubbles when we heat water, or water carried along in a steam pipe
- How do we define properties for such a mixed system? We have neither an incompressible liquid nor an ideal gas

Phase Transitions

- Consider if we had a tank with a liquid and a vapour
  - The molecules in the liquid are much closer together; in the vapour molecules are much further apart
  - The liquid molecules have varying energies, some with enough energy to break loose (evaporation)
  - At the same time, the vapour molecules are hitting the surface of the liquid, some with low enough energy that they are re-captured (condensation)
  - Eventually we will reach a point where the rate of evaporation equals the rate of condensation, and we have *phase equilibrium*
- At phase equilibrium, the pressure of the system is constant; we define this as the *saturation pressure* P_{sat}
  - The temperature will also be constant, defined as the *saturation temperature* T_{sat}
- If we increase the temperature, more molecules end up in the vapour at phase equilibrium ($P_{sat}, T_{sat} \uparrow$)
- The rate of evaporation is proportional to the surface area
- If we consider the energy distribution of the molecules, the molecules evaporating have a higher energy, so evaporation removes the most energetic molecules, resulting in cooling of a liquid
- To keep the temperature constant, we need to supply energy
  - This is defined as the *latent heat of evaporation*
- Consider a system with a liquid and vapour and consider an infinitesimal amount of mass δm going from liquid to vapour
  - Treat either the liquid or the vapour as an open system
  - How much is the energy change of this system?
  - Consider forcing the system to a constant temperature and pressure
  - Consider an infinitesimal δs being pushed into the system
  - The net change in internal energy (PE, KE change assumed 0) so $\mathrm{d}u = \delta q + \delta w + h\,\mathrm{d}m$
    - Enthalpy change includes flow rate
  - Entropy change is $\mathrm{d}s = \dfrac{\delta q}{T} + \delta \mathrm{d}m \implies \delta q = T\,\mathrm{d}s - Ts\,\mathrm{d}m$
  - If we assume reversibility $\delta w = -P\,\mathrm{d}V$
  - $\mathrm{d}U = T\mathrm{d}s - P\,\mathrm{d}V + (h - Ts)\,\mathrm{d}m$
- Energy change is due to 3 terms: heat transfer, work, and mass transfer

Definition

The Gibbs Energy $G = H - TS$ and the specific Gibbs Energy $g = \dfrac{G}{m} = h - Ts$, also known as the chemical potential

A difference in the Gibbs energy results in mass transfer

- Then energy change is $\mathrm{d}U = T\,\mathrm{d}s - P\,\mathrm{d}V + g\,\mathrm{d}m$
- g is the increase in energy of an open system per unit mass added

Important

Gibbs equation for an open system:

$$\mathrm{d}s = \frac{\mathrm{d}U}{T} + \frac{P}{T}\mathrm{d}V - \frac{g}{T}\,\mathrm{d}m$$

Phase Equilibrium

- Consider an isolated system partially filled with saturated liquid and partially filled with saturated vapour
- What is the condition for phase equilibrium?
  - Start with the state principle: $S = S(U, V, m)$
    - We have two components, the saturated vapour (subscript g) and the saturated liquid (subscript f)
  - Total entropy: $S = S_f(U_f, V_f, m_f) + S_g(U_g, V_g, m_g)$
  - At equilibrium $\mathrm{d}S = 0 \implies \mathrm{d}S_f + \mathrm{d}S_g = 0$
  - $\mathrm{d}S_f = \frac{\mathrm{d}U_f}{T_f} + \frac{P_f}{T_f}\mathrm{d}V_f - \frac{g_f}{T_f}\mathrm{d}m_f, \mathrm{d}S_g = \frac{\mathrm{d}U_g}{T_g} + \frac{P_g}{T_g}\mathrm{d}V_g - \frac{g_g}{T_g}\mathrm{d}m_g$
  - Isolated system so $\mathrm{d}U_f = -\mathrm{d}U_g, \mathrm{d}V_f = -\mathrm{d}V_g, \mathrm{d}m_f = -\mathrm{d}m_g$
  - Combining these two we get $\mathrm{d}S = \left(\frac{1}{T_f} - \frac{1}{T_g}\right)\mathrm{d}U_f + \left(\frac{P_f}{T_F} - \frac{P_g}{T_g}\right)\mathrm{d}V_f + \left(\frac{g_f}{T_f} - \frac{g_g}{T_g}\right)\mathrm{d}m_f = 0$
  - $\mathrm{d}U_f, \mathrm{d}V_f, \mathrm{d}m_f$ are all arbitrary (we never said how much the mass, volume, etc had to change), so each of the terms in the brackets are independently zero
  - Therefore the equilibrium condition is: $\begin{cases} T_f = T_g \\ P_f = P_g \\ g_f = g_g \end{cases}$
- $g_f = g_g$ is new – at equilibrium there is no exchange of gas between two phases whose chemical potentials are the same
- g, the specific Gibbs energy, drives mass exchange
  - Assume T, P are constant but $g_f > g_g$, then $\mathrm{d}S = -\frac{1}{T_f}(g_f - g_g)\mathrm{d}m_f > 0 \implies \mathrm{d}m_f < 0$
  - If $g_f > g_g$ then there is mass transfer from the liquid to the vapour (evaporation)
  - Conversely if $g_g < g_f$ then there is mass transfer from vapour to liquid (condensation)
- If chemical potentials are unequal, then there will be mass exchange until chemical potentials are equal

Lecture 14, Oct 11, 2022

Gibbs-Duhem Equation

- How do we evaluate g in terms of known quantities?
  - We want $g(T, P)$ because in the end these are the only things we can directly measure
- $G = H - TS = U + PV - TS \implies \mathrm{d}G = \mathrm{d}U + P\mathrm{d}V + V\mathrm{d}P - T\mathrm{d}S - S\mathrm{d}T$
  - Using the Gibbs equation $\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V + g\mathrm{d}m$ and substitute this in
  - $\mathrm{d}G = T\mathrm{d}S - P\mathrm{d}V + g\mathrm{d}m + P\mathrm{d}V + V\mathrm{d}P - T\mathrm{d}S - S\mathrm{d}T = g\mathrm{d}m + V\mathrm{d}P - S\mathrm{d}T$
  - Since $G = gm \implies \mathrm{d}G = g\mathrm{d}m + m\mathrm{d}g \implies g\mathrm{d}m + m\mathrm{d}g = g\mathrm{d}m + V\mathrm{d}P - S\mathrm{d}T$
  - $m\mathrm{d}g = V\mathrm{d}P - S\mathrm{d}T \implies \mathrm{d}g = v\mathrm{d}P - s\mathrm{d}P$, known as the Gibbs-Duhem Equation

Definition

The Gibbs-Duhem Equation: $\mathrm{d}g = v\mathrm{d}P - s\mathrm{d}T$

Phase Equilibrium

- Intuitively as we heat the system the pressure should increase
  - Plotting P vs T gives us a line; every point along this line is an equilibrium state
  - For every P_{sat} saturation pressure we have a corresponding saturation temperature T_{sat}
  - Can we derive this line?
- At equilibrium $g_f = g_g$; suppose we change temperature by $\mathrm{d}T$ and pressure by $\mathrm{d}P$ and get a new equilibrium
  - At the new equilibrium $g_f + \mathrm{d}g_f = g_g + \mathrm{d}g_g \implies \mathrm{d}g_f = \mathrm{d}g_g$
  - Using the Gibbs-Duhem equation: $v_f \mathrm{d}P - s_f \mathrm{d}T = v_g \mathrm{d}P - s_g \mathrm{d}T$
    - Note $\mathrm{d}P$ and $\mathrm{d}T$ are the same for the liquid and gas
  - $\dfrac{\mathrm{d}P}{\mathrm{d}T} = \dfrac{s_g - s_f}{v_g - v_f}$ is the slope of our P-T curve, however s_f, s_g are hard to determine
    - It's easier to work with h because it's the latent heat, which can be looked up
- We now want a relationship between h and s
  - $h = u + Pv \implies \mathrm{d}h = \mathrm{d}u + P\mathrm{d}v + v\mathrm{d}P$
  - Using the Gibbs equation: $T\mathrm{d}s = \mathrm{d}u + P\mathrm{d}v \implies \mathrm{d}h = T\mathrm{d}s + v\mathrm{d}P$
  - Since the system is in equilibrium, $\mathrm{d}P = \mathrm{d}v = 0$ and $\mathrm{d}h = T\mathrm{d}s, T = T_{sat}, P = P_{sat}$
  - $\mathrm{d}s = \dfrac{\mathrm{d}h}{T} \implies s_2 - s_1 = \displaystyle\int \frac{\mathrm{d}h}{T} = \frac{h_2 - h_1}{T}$ since T is a constant
    - This makes sense because $Q_{12} = h_2 - h_1$, because entropy generation in this system can only be due to heat added, which goes directly to enthalpy

Definition

The latent heat of vaporization $h_{fg} = h_g - h_f$

This can be looked up in a table for specific substances

Definition

The Clapeyron Equation (applies to any two phases): $\dfrac{\mathrm{d}P}{\mathrm{d}T} = \dfrac{h_{fg}}{T(v_g - v_f)}$

- Note we never made any assumptions about the phases, so this equation applies for any two phases (e.g. liquid-solid, gas-solid)

Liquid-Vapour Equilibrium

- If we assume that this is a liquid and gas, then we can assume $v_f \ll v_g$
- Also assume the vapour is an ideal gas, so $v_g = \dfrac{RT}{P}$
- Putting this back into the equation we get $\dfrac{\mathrm{d}P}{\mathrm{d}T} = \dfrac{h_{fg}P}{RT^2}$
- Assume h_{fg} is constant, then $\displaystyle\int \frac{\mathrm{d}P}{P} = \frac{h_{fg}}{R} \int \frac{\mathrm{d}T}{T^2}$
- $\ln P = -\dfrac{h_{fg}}{RT} + C$ is the Clausius-Clapeyron Equation

Definition

The Clausius-Clapeyron Equation (only applies for liquid-vapour systems): $\ln P = -\dfrac{h_{fg}}{RT} + C$

- This means $P_{sat} = c \exp\left(-\dfrac{h_{fg}}{RT_{sat}}\right)$

- If we know one state, we can calculate the entire curve
  - e.g. for water at $P_{sat} = 1\text{atm}$, $T_{sat} = 100°\text{C}$, so we can calculate c to get the entire curve
- Example: in a pressure cooker P_{sat} is high, so T_{sat} is also high and cooks the food faster

Solid-Liquid Equilibrium

- Denote solid phase by subscript s, so $\dfrac{\mathrm{d}P}{\mathrm{d}T} = \dfrac{h_{sf}}{T(v_f - v_s)}$ where h_{sf} is the latent heat of fusion
  - Is this slope greater or less than zero?
  - $v_f - v_s$ is the only thing that can be negative
- Suppose we have a liquid and we put a solid in it, which sinks to the bottom, then $\rho_s > \rho_f \implies v_s < v_f$ and $v_f - v_s > 0$
  - For most solids, they are denser than their liquid counterparts, so $\dfrac{\mathrm{d}P}{\mathrm{d}T} > 0$
- One big exception is water, which has $\rho_f > \rho_s \implies \dfrac{\mathrm{d}P}{\mathrm{d}T} < 0$
- For most solids, the separation between solid and liquid and liquid and vapour both have positive slopes on a P-T diagram
  - The solid-vapour boundary also has a positive slope
- We can have phase transitions across any of the 3 boundaries
- The point at which all 3 boundaries meet is the *triple point*, the point where liquid, solid, and vapour are in equilibrium
  - At the triple point, we can have boiling and freezing at the same time!

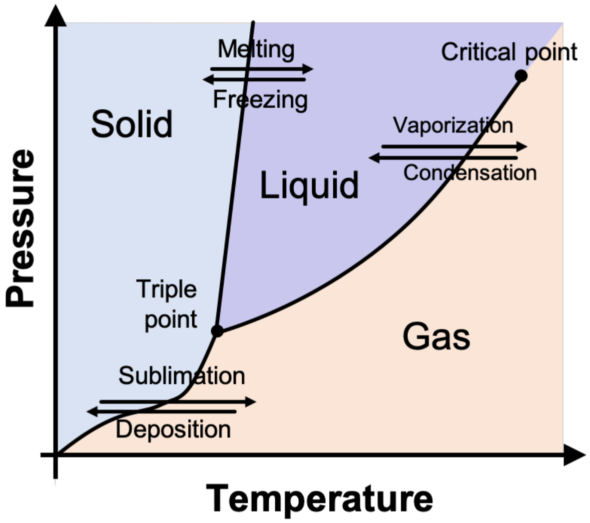

Figure 2: Phase diagram with triple point

- For water this is at $(0.01°\text{C}, 61.73\text{Pa})$
- The triple point is *invariant*, so we can use it for calibration of instruments

- For water/ice, if we increase the pressure, the melting point decreases
  - This is why if we apply pressure on ice, the ice around that pressure point melts (regelation)
  - This is also how skates work
- If we want to dry something, we can either increase the temperature (oven drying), or decrease the pressure (vacuum drying)
  - However both of these methods don't work well for delicate things like cells
  - Freeze drying decreases the temperature first, and then decrease the pressure; the moisture sublimates directly and delicate structures are preserved due to the freezing

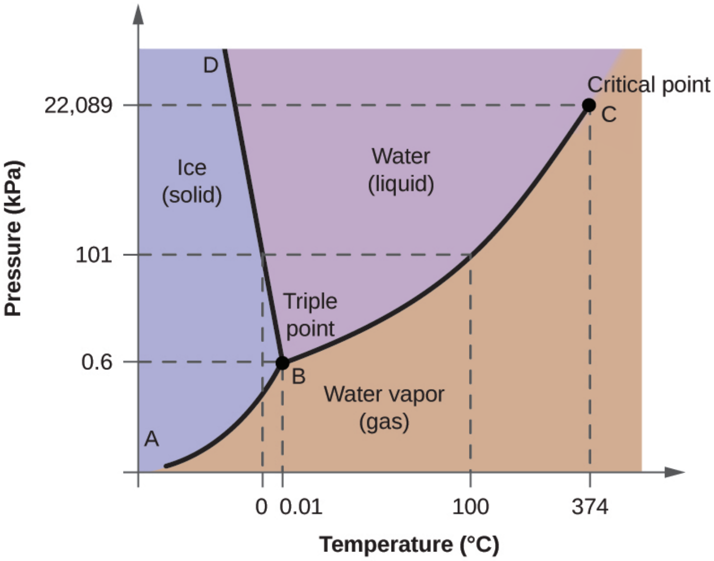

Figure 3: Phase diagram of water

Lecture 15, Oct 13, 2022

Phase Change in Constant Pressure Systems

- Consider a system with a liquid with temperature T and volume V; heat the system under constant pressure
  - Eventually we will reach T_{sat}, and the system has a combination of a vapour and liquid
  - Eventually all the liquid changes into vapour, and the vapour expands as an ideal gas
  - A plot of T against v will show an increasing line up to T_{sat}, then the temperature stays constant, but volume increases substantially, until all the liquid changes into vapour and the temperature increases again
- v_f is the specific volume of the saturated liquid, v at T_{sat}; v_g is the specific volume of the saturated vapour, v when all the liquid changes into vapour
  - A higher pressure increases v_f but decreases v_g
  - If we keep increasing pressure, these will eventually meet; this is known as the *critical pressure*
  - At the critical pressure we no longer have a horizontal line in the middle but just an inflection point
- If we draw a line through all the v_f, v_g for various pressures, we get the *vapour dome*
- Where $T < T_{sat}$ we have a *subcooled liquid*; where $T > T_{sat}$ we have a *superheated vapour*
  - In the middle, the horizontal line where $T = T_{sat}$, we have the *saturated mixture*
- Since $v_f = v_g$ at the critical point, we can regard it as one phase, as a *supercritical fluid*
  - Visually we no longer see any boundaries

Phase Change in Constant Temperature Systems

- Consider a system with only vapour at constant temperature; if we compress this system, we will get some liquid forming, and then eventually only liquid
- On a P-v diagram:
- We need two independent intensive properties to define the state of a system
  - If we have a mixture, the temperature and pressure are not independent, they are related by the Clausius-Clapeyron Equation (e.g. if we have boiling water at 1atm, we immediately know the pressure)

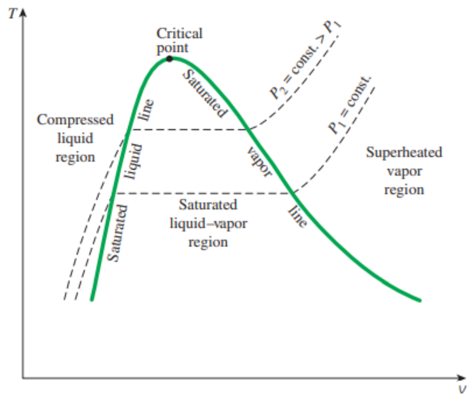

Figure 4: Vapour dome on a T-v diagram

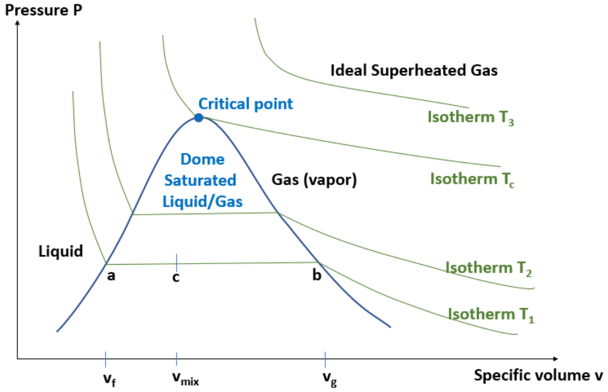

Figure 5: Phase change in a constant temperature system

    - Therefore we need one additional property to fix the state for mixtures
- Define a new property, the *quality* $x = \frac{m_g}{m}$ where m_g is the mass of the vapour and m is the mass of the mixture
    - $x = 0 \implies$ saturated liquid, $x = 1 \implies$ saturated vapour, $0 < x < 1$ is a saturated mixture
- The quality can be used to determine the volume
- Suppose the volume of a mixture is $V = mv = m_g v_g + m_f v_f \implies v = \frac{m_g}{m} v_g + \frac{m_f}{m} v_f$
    - $v = x v_g + (1 - x) v_f$
    - This works for any other property – $h = x h_g + (1 - x) h_f = h_f + x(h_g - h_f) = h_f + x h_{fg}$ where h_{fg} is the latent heat of vaporization

Lecture 16, Oct 17, 2022

Heat Engines

- What is the most efficient engine possible?
- A heat engine is any device that works in a thermodynamic cycle which does work on its surroundings as long as heat is supplied
    - The engine must do a net amount of work (that is, it can use work internally, but it must produce work output)
    - It must be able to work continuously (so energy storage devices do not count)
- We can model a heat as a system that takes heat Q_H from a hot thermal reservoir T_H, puts heat Q_C into a cold thermal reservoir T_C while producing work W_{net}
- Define the thermal efficiency as $\eta_{th} = \frac{W_{net}}{Q_H}$ (neglecting sign convention)
    - So what is the maximum possible efficiency?
- A note on perpetual motion machines:
    - PMM of the first kind violate the first law (they create energy from nothing)
    - PMM of the second kind violate the second law
        - This is a little more subtle to see
        - e.g. an engine that directly converts heat to work, without the use of a cold thermal reservoir
            - Over a thermodynamic cycle, all properties go to their initial values
            - However in this engine, heat only comes in, so the entropy can only increase and is never removed from the system, making it an invalid heat engine
            - This is why heat engines must have a cold reservoir to reject heat in order to remove entropy
- Kevin-Planck statement: It is impossible for any device operating in a thermodynamic cycle to receive heat from a high temperature source and produce work without rejecting heat to a low temperature sink
    - This can be an alternative statement of the second law

Carnot Engine

- A Carnot engine is the theoretically most efficient heat engine possible
- Consider the same system that takes heat Q_H from a hot thermal reservoir T_H, puts heat Q_C into a cold thermal reservoir T_C while producing work W_{net}
- All processes are reversible; no s_{gen}, the engine is frictionless, and all heat and work transfer is reversible
- Over a cycle $\Delta E = 0 \implies Q_H - Q_C - W = 0 \implies W = Q_H - Q_C$
- $\Delta S = 0 \implies \oint \frac{\delta Q_{rev}}{T} = 0$; this can be broken down into the entropy from heat addition and heat rejection
- We want to minimize the entropy added during heat addition, because we need to get rid of it later
    - This means we need to maximize the temperature, so add heat when $T = T_H$, which means the heat transfer is isothermal

  - $\Delta S_{\text{heat addition}} = \frac{Q_H}{T_H}$
- We also want to maximize the entropy removed during heat rejection
  - This means we need to reject heat at the lowest temperature, so reject heat at $T = T_C$
  - $\Delta S_{\text{heat rejection}} = \frac{Q_C}{T_C}$
- To close the thermodynamic cycle, temperature change is done first through an isentropic expansion, then an isentropic compression

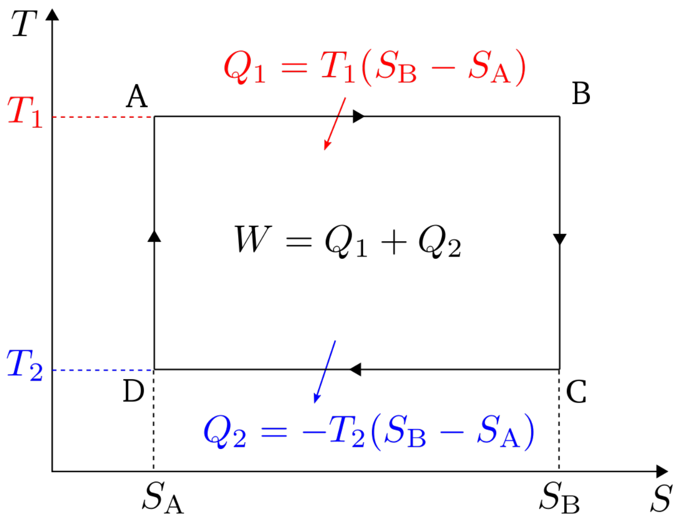

Figure 6: Carnot engine on a T-S diagram

- $\Delta S_{\text{heat addition}} - \Delta S_{\text{heat rejection}} = 0 \implies \frac{Q_H}{T_H} = \frac{Q_C}{T_C}$ and $W_{net} = Q_H - Q_C$
  - $\eta_{th} = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H} = 1 - \frac{T_C}{T_H}$

Definition

The Carnot Efficiency: $\eta_{th} = 1 - \frac{T_C}{T_H}$ is the theoretical maximum efficiency of a heat engine

- This means the maximum efficiency of an engine depends only on the temperatures of the thermal reservoirs and not the engine cycle
  - T_C is usually fixed, so the higher we raise T_H, the better our efficiency

Implementing a Carnot Cycle

- Practically we can do a two-phase Carnot cycle
- Isothermal heat addition can be accomplished with a boiler
  - If we find a liquid that boils at $T_H - \Delta T$, we can pipe the liquid into the boiler, which takes heat from the heat source and the liquid emerges as vapour
- The resulting vapour can be passed into an isentropic turbine, which cools the liquid, extracting work and producing a liquid-vapour mixture
- The heat rejection can be accomplished with a condenser
  - The liquid-vapour mixture would be at temperature $T_C + \Delta T$, which rejects heat to a cold heat sink and produces a liquid
- Finally the liquid is is passed into an isentropic compressor (which requires work, but can be driven by the turbine) and the cycle is complete

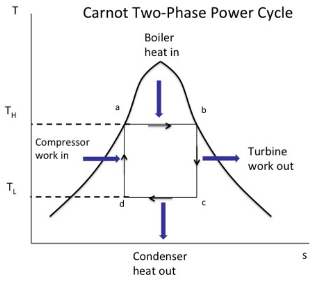

Figure 7: Two-phase Carnot cycle on a T-S diagram

Lecture 17, Oct 18, 2022

Refrigerators

- A refrigerator is a device that takes heat from a low-temperature region to a higher temperature region, when being supplied with work
  - Essentially a heat engine working in reverse
- Heat Q_C is taken from the cold reservoir T_C and heat Q_H is rejected to the hot reservoir T_H, while work W is going in
- A liquid absorbs latent heat when it evaporates, and rejects heat when condensing
  - Using a low-pressure evaporator and high-pressure compressor, we can make the condensation temperature higher than the evaporation temperature since T_{sat} is dependent on P

Carnot Refrigerator

- A Carnot refrigerator is the theoretical maximumly efficient refrigerator
- Condenser $T = T_H + \Delta T$ feeds into a turbine, then an evaporator with $T = T_C - \Delta T$, then a compressor and back into the condenser

Definition

The Coefficient of Performance for a refrigerator: $\text{cop}_R = \dfrac{Q_C}{W_{net}}$

- Energy balance: $W_{net} = Q_H - Q_C \implies \text{cop}_R = \dfrac{Q_C}{Q_H - Q_C} = \dfrac{1}{\frac{Q_H}{Q_C} - 1}$
- Entropy balance for Carnot cycle: $\dfrac{Q_H}{T_H} - \dfrac{Q_C}{T_C} = 0 \implies \dfrac{Q_H}{Q_C} = \dfrac{T_H}{T_C}$
- Therefore $\text{cop}_R = \dfrac{1}{\frac{T_H}{T_C} - 1}$
  - Since $T_H > T_C$, $\text{cop}_R > 0$, i.e. you always need to supply work to a refrigerator
    - This is stated in the Clausius statement
  - Typically it is greater than 1
    - For typical domestic refrigeration it is 2-3
  - Note $\lim\limits_{T_H \to T_C} \text{cop}_R = \infty$; i.e. the smaller the temperature difference, the better the performance
- Notice that as $T_C \to 0, \text{cop}_R \to 0$, so as T_C approaches absolute zero, we need more and more work; we can never reach absolute zero because that would require infinite work
  - This is the counterpart to how thermal efficiency $\eta_{th} \to \infty$ as $T_C \to 0$

- Heat pumps use refrigerators as heating devices
  - The heat added is $Q_H = Q_C + W_{net}$ for heat pumps while regular heating has $Q_H = W_{net}$, so heat pumps are significantly more efficient
- Define the coefficient of performance for a heat pump to be $\text{cop}_{HP} = \dfrac{Q_H}{W_{net}}$, which works out to $\dfrac{1}{1 - \frac{T_C}{T_H}}$
  - Since $\dfrac{T_C}{T_H} < 1$, $\text{cop}_{HP} > 1$
  - As $T_C \to T_H$, the performance increases – this is why heat pumps are less effective in colder climates
  - A solution is ground-coupled heat pumps, which draw heat from inside the ground instead of the outside air
- We can design refrigerators/heat pumps such that they can be easily reversed, so in the summer we can use it as an AC, and in the winter as a heater

Carnot Principles

1. The efficiency of a reversible heat engine is always greater than that of an irreversible engine operating between the same temperatures
   - Consider two engines, one reversible and one irreversible, connected to the same temperatures
   - If work from the reversible engine, W_R, is less than the work from the irreversible engine, W_I, then $Q_{C,I} < Q_{C,R}$
   - Consider if we ran the reversible engine as a refrigerator, and put the resulting heat into the irreversible engine, and use the irreversible engine to drive the reversible engine
   - This gives us a device that interacts only with one thermal reservoir and gives us work directly, which makes it a perpetual motion machine
2. The efficiency of all reversible engines operating between the same two temperatures are the same
   - Efficiency depends only on temperatures, not engine design, in a reversible engine
   - This can be proven in exactly the same way

Lecture 18, Oct 20, 2022

Heat Transfer

- Thermodynamics typically only deals with equilibrium/quasi-equilibrium processes; in reality we care about kinetics (i.e. we need to worry about time), so the rate of heat transfer matters
- In heat transfer we deal with a lot of rates
- Various applications:
  - Insulating buildings, HVAC systems
  - Electronics cooling
  - Manufacturing/industry

Mechanisms of Heat Transfer

1. Conduction
   - Transfer of heat through a stationary medium, e.g. heat transfer through a metal, heat transfer through a window with an air gap
   - Driven by a temperature difference between two points in a stationary medium
2. Convection
   - Transfer of heat between a solid and surface and adjacent fluid that flows, e.g. moving air across a hot plate to cool it
   - Actually a combination of fluid mechanics and heat conduction
3. Radiation
   - Energy emitted by matter, e.g. the sun
   - Unlike the other forms, radiation can pass through a vacuum

Heat Flux

- Defined as the heat transfer rate per unit area
- Flux is defined as $\dfrac{\dot{Q}}{A}$, with units of heat transfer rate per unit area (W/cm^2)
- How do we reduce heat flux?
    - Reduce $\dot{Q}$ (less heat generation) – oftentimes not possible
    - Increase A (more area to dissipate heat)
- Size is important – due to the square-cube law, the specific surface area $\dfrac{A}{V}$ goes down as an object gets larger
    - If heat generation is proportional to volume, now we have less area to transfer heat per unit of heat generation
    - Larger systems are generally harder to cool if heat generation is proportional to volume
- Heat sinks are designed to maximally increase surface area
- We can also increase the amount of airflow (increasing the effects of convection)
- Some fluids are more effective at convection, e.g. water cooling

Lecture 19, Oct 24, 2022

Conduction

- Heat transfer through a stationary medium as a result of a temperature difference
- Temperature T_1 on one side and T_2 on the other, resulting in a heat transfer $\dot{Q}$
- Conduction can occur in any material – solid, liquid or gas, but without long range motion in the medium
- “Thermometers are speedometers for atoms”
    - In a solid, atoms can vibrate in their lattice
    - In a fluid, atoms and molecules can translate, molecules can vibrate and rotate
    - At higher temperature, atoms move faster; collisions between them transfer energy, which is heat transfer
- Conduction is modelled by *Fourier’s Law*
    - Consider a temperature gradient T_1, T_2; somewhere in the middle we have T_0
    - Heat flux $\dot{q} = \dfrac{\dot{Q}}{A}$, with $\dot{q}^+$ from T_0 to T_2 and $\dot{q}^-$ from T_0 to T_1
    - Define the average molecular velocity $\bar{v}$, number density n, and mean free path λ
        - Over a distance of λ the direction of the molecules should be constant
    - Molecular energy is $mcT(y)$ where m is the molecular mass, c is the specific heat capacity and $T(y)$ is temperature
    - The heat flux is defined as the number of molecules crossing a unit area per unit time, times the number energy per molecule
        - $\dot{q} = n\bar{v}mcT(y)$
        - The positive heat flux is $\dot{q}^+ = n\bar{v}mcT\left(-\dfrac{\lambda}{2}\right)$
        - The negative flux is $\dot{q}^- = n\bar{v}mcT\left(\dfrac{\lambda}{2}\right)$
        - The net heat flux is the difference, $n\bar{v}mc\left(T\left(-\dfrac{\lambda}{2}\right) - T\left(\dfrac{\lambda}{2}\right)\right)$
        - We can approximate $T\left(-\dfrac{\lambda}{2}\right) - T\left(\dfrac{\lambda}{2}\right)$ as $-\lambda\dfrac{\mathrm{d}T}{\mathrm{d}y}$
    - $\dot{q} = -n\bar{v}mc\lambda\dfrac{\mathrm{d}T}{\mathrm{d}y}$
    - The first 5 constants are all properties of the gas, which we define to be k, the thermal conductivity

Equation

Fourier's Law:

$$\dot{Q} = \dot{q}A = -kA\frac{\mathrm{d}T}{\mathrm{d}y}$$

where the thermal conductivity $k = n\bar{v}mc\lambda$, where n is the number density, $\bar{v}$ is the velocity, m is the mass per molecule, c is the heat capacity and λ is the mean free path

In reality $k = k(T, P)$, but we assume it to be constant

- Notes:
  - k has units of W$m.K$
  - The sign is negative – the direction of heat conduction is opposite to the temperature gradient
  - The closer the atoms are, the better the thermal conductivity
    - Typically k is the greatest for solids, then liquids, then gases
- In addition to transfer of kinetic energy, conduction can also occur through electron flow
  - This is why good electrical conductors are usually good thermal conductors

Convection

- Heat transfer between a solid surface and a moving fluid
- Two heat transfer mechanisms:
  1. Motion/collision of fluid molecules (conduction)
  2. Energy transfer due to bulk motion of fluid (advection)
- We care about fluid flow near surfaces
  - Near the surface of the solid, we have the *boundary layer*, where the fluid is severely slowed down
  - At the surface we have a no-slip condition, i.e. the fluid has zero velocity
  - The edge of the boundary later is defined as where $v = 0.99v_{free}$ where v_{free} is the free stream velocity
- When Re < 2000, the flow is laminar
- Temperature will also have a boundary layer – the temperature as you approach the surface differers from that of the bulk
  - $\dot{Q}_{conv} = -kA\frac{\mathrm{d}T}{\mathrm{d}y}$, but $\frac{\mathrm{d}T}{\mathrm{d}y}$ is a complex function of fluid mechanics
  - We typically use Newton's Law of Cooling, $\dot{Q}_{conv} = hA(T_s - T_\infty)$, where h is the heat transfer coefficient in units of $\mathrm{W/m^2\,K}$
    - h is a function of fluid flow and properties
      - Thermal conductivity of the fluid
      - Surface geometry
      - Fluid velocity (higher average velocity leads to a higher h)

Equation

Newton's Law of Cooling: $\dot{Q}_{conv} = hA(T_s - T_\infty)$

Lecture 20, Oct 25, 2022

Radiation

- Energy emitted by all matter in the form of electromagnetic waves
  - Thermal radiation is emitted by all bodies above absolute zero
- Typically volumetric, i.e. scales with volume of the body

  - However for opaque objects, radiation can only be emitted from the surface, so radiation scales with surface area
- Amount of radiation is a function of the surface temperature

Equation

Stefan-Boltzmann Law: The maximum amount of radiation that a surface can emit is

$$\dot{Q}_{max} = \sigma A T_s^4$$

where A is the surface area, T_s is the surface temperature, and σ is the Stefan-Boltzmann constant, $\sigma = 5.67 \times 10^{-8}\,\mathrm{W/m^2\,K^4}$

- For a blackbody, the higher the temperature, the more the distribution of wavelengths shifts towards shorter wavelengths
  - A blackbody is a surface that emits the maximum amount of radiation
  - However real surfaces emit less radiation
- For a real surface, $\dot{Q} = \varepsilon\sigma A T^4$, where ε is the emissivity, $0 \leq \varepsilon \leq 1$
  - When $\varepsilon = 1$, the body is a blackbody
  - Otherwise it is a *graybody*
- Real objects have complex wavelength distributions that can be approximated by a graybody
  - A graybody has a constant emissivity less than 1
- e.g. the emissivity of black paint is 0.99; aluminum foil has an emissivity of 0.07
- Example: Liquid N_2 is kept inside a vacuum Dewar Flask
  - There are 2 layers of glass separated by a vacuum to prevent conduction and convection
  - The surfaces are coated with silver, which has a very low emissivity, to prevent radiation
- Radiation can also be absorbed when it's incident on a surface
  - Some radiation is absorbed and some is reflected for an opaque system
  - For a blackbody everything is absorbed
  - The fraction absorbed is defined as the *absorptivity* α, such that $\dot{Q}_{\text{absorbed}} = \alpha\dot{Q}_{\text{incident}}$
  - From conservation of energy, the amount reflected is $\dot{Q}_{\text{reflected}} = (1-\alpha)\dot{Q}_{\text{incident}}$
  - Kirchhoff's Law: $\alpha = \varepsilon$
- Special case: when a small surface is completely surrounded by a much larger surface, $\dot{Q}_{net} = \varepsilon\sigma A_s(T_s^4 - T_{surr}^4)$
  - This net radiation accounts for radiation absorbed and emitted
- Example: Chip with $\varepsilon = 0.6$ in a room with air/wall temperature 25°C, and $A_s = (0.015\mathrm{m})^2$
  - Natural (free) convection
    - Estimate using a simple model $h = c(T_s - T_\infty)^{\frac{1}{4}}$ where $c = 4.2\,\mathrm{W/m^2\,K^{5/4}}$
  - Forced convection with $h = 250\mathrm{W/m^2\,K}$
  - What is the maximum power we can dissipate if the chip temperature must be less than 85°C?
    - Natural convection: $\dot{Q}_{conv} = hA(T_s - T_\infty) = cA(T_s - T_\infty)^{\frac{1}{4}}(T_s - T_\infty) = 0.158\mathrm{W}$
    - Radiation: $\dot{Q}_{rad} = \varepsilon A\sigma(T_s^4 - T_{surr}^4) = 0.065\mathrm{W}$
    - Net heat transfer: $\dot{Q} = \dot{Q}_{conv} + \dot{Q}_{rad} = 0.223\mathrm{W}$
      - Note here the convective and radiative heat transfers are of similar magnitude
        - Typically values for free convection is 3-20 $\mathrm{W/m^2\,K}$
    - Forced convection: $\dot{Q}_{conv} = 3.375\mathrm{W}$
      - In this case the radiative heat transfer is only about 2% of the total heat transfer, so we can ignore it

Summary

For heat radiation:

- For an ideal blackbody:
$$\dot{Q} = \sigma A T^4$$
- For a real object:
$$\dot{Q} = \varepsilon \sigma A T^4$$
- If a smaller surface is surrounded by a larger surface:
$$\dot{Q}_{net} = \varepsilon \sigma A_s (T_s^4 - T_{surr}^4)$$

where $\sigma = 5.67 \times 10^{-8}\,\mathrm{W/m^2K^4}$ and ε is the surface emissivity, which is equal to α, the surface absorptivity

Thermal Conductivity Via Electrons

- Wiedemann-Franz Law: relates thermal and electrical conductivities of metals: $\frac{k}{\sigma} = LT$ where σ is the electrical conductivity, k is the thermal conductivity, L is the Lorenz number $L = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2 = 2.44 \times 10^{-8}\,\mathrm{V^2/K^2}$

Heat Conduction

- Consider system volume, we can have 3D heat conduction $\dot{Q}_x, \dot{Q}_y, \dot{Q}_z$
  - Temperature is a function of position and time
  - Heat conduction is a vector, $\vec{\dot{Q}} = \dot{Q}_x\hat{\imath} + \dot{Q}_y\hat{\jmath} + \dot{Q}_z\hat{k}$
  - Use Fourier's law: $\dot{Q}_x = -kA_x\frac{\mathrm{d}T}{\mathrm{d}x}, \dot{Q}_y = -kA_y\frac{\mathrm{d}T}{\mathrm{d}y}, \dot{Q}_z = -kA_z\frac{\mathrm{d}T}{\mathrm{d}z}$
- We will assume our system is 1 dimensional
  - This can happen if $\Delta x \ll \Delta y, \Delta z$ (e.g. a wall or through a plate), so $\frac{\mathrm{d}T}{\mathrm{d}x} \gg \frac{\mathrm{d}T}{\mathrm{d}y}, \frac{\mathrm{d}T}{\mathrm{d}z}$ and we can ignore the other two dimensions
  - This can also happen if we have insulated sides
  - Can also happen in cylindrical coordinates

Lecture 21, Oct 27, 2022

One Dimensional Heat Conduction Equation

- Consider heat conduction $x \to x + \Delta x$, surface area at x is A_x; what is the temperature as a function of x?
- $\frac{\mathrm{d}h}{\mathrm{d}t} = mc_p\frac{\partial T}{\partial t} = \rho V c_p\frac{\partial T}{\partial t} = \rho c_p A\mathrm{d}x\frac{\partial T}{\partial t}$
- Energy balance: $\rho c_p A\mathrm{d}x\frac{\partial T}{\partial t} = \dot{Q}_x - \dot{Q}_{x+\Delta x}$
  - In terms of heat flux, $\dot{q}_x A_x - \dot{q}_{x+\Delta x}A_{x+\Delta x}$
  - $\rho c_p\frac{\partial T}{\partial t} = -\frac{1}{A}\left(\frac{\dot{q}_x A_x - \dot{q}_{x+\Delta x}A_{x+\Delta x}}{\Delta x}\right)$
  - Take the limit $\Delta x \to 0$: $\rho c_p\frac{\partial T}{\partial t} = -\frac{1}{A}\frac{\partial(\dot{q}A)}{\partial x}$

Cartesian Coordinates

- Consider Cartesian coordinates, constant area, then we can simplify this as $\rho c_p \frac{\partial T}{\partial t} = -\frac{\partial \dot{q}}{\partial x}$
- Putting this into Fourier's law, $\dot{q} = -k\frac{\mathrm{d}T}{\mathrm{d}x}$, we get $\rho c_p \frac{\partial T}{\partial t} = -\frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x}\right)$
  - With a constant k, we get $\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$
  - Alternatively $\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} = \alpha\frac{\partial^2 T}{\partial x^2}$
- $\alpha = \frac{k}{\rho c_p}$ is the thermal diffusivity, with units of m^2/s
  - Higher k conducts heat well so the gradient is sharper
  - ρc_p stores energy well, so a lot of heat can enter the system without changing the temperature much
  - $\alpha = 1.11 \times 10^{-4}\,\mathrm{m}^2/\mathrm{s}$ for copper, $\alpha = 3.4 \times 10^{-7}\,\mathrm{m}^2/\mathrm{s}$
- For steady state, $\frac{\partial T}{\partial t} = 0$ so $\alpha\frac{\partial^2 T}{\partial x^2} = 0$
  - Integrate this and we get that $\frac{\partial T}{\partial x}$ is a constant

Cylindrical Coordinates

- In the radial direction Fourier's law is $\dot{q} = -k\frac{\partial T}{\partial r}$
- $A = 2\pi r L$
- $\rho c_p \frac{\partial T}{\partial t} = -\frac{1}{2\pi r L}\left(\frac{\partial}{\partial r}\left(2\pi r L\left(-k\frac{\partial T}{\partial r}\right)\right)\right) = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$
- $\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$
- For steady state, this simplifies to $\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0 \implies r\frac{\partial T}{\partial r} = 0$

Spherical Coordinates

- $A = 4\pi r^2$, $\dot{q} = -k\frac{\partial T}{\partial r}$
- Doing the same derivation gets us $\rho c_p \frac{\partial T}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$
- $\frac{\partial T}{\partial t} = \frac{\alpha}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$

Summary

1D heat flow equations:

- Cartesian coordinates:
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$
- Cylindrical coordinates (radial):
$$\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$
- Spherical coordinates (radial):
$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$$

where α is the thermal diffusivity, $\alpha = \frac{k}{\rho c_p}$

In general,
$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial T}{\partial r}\right)$$
where $n = 0$ for Cartesian, $n = 1$ for cylindrical and $n = 2$ for spherical

Lecture 22, Oct 31, 2022

Thermal Resistance

- Recall in steady-state 1D heat flow we derived $\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = 0 \implies T(x) = \frac{T_2 - T_1}{L}x + T_1$
    - $\dot{q} = -k\frac{\mathrm{d}T}{\mathrm{d}x} = -\frac{k}{L}(T_2 - T_1)$
    - Other boundary conditions:
        1. Known $\dot{q}_1, T_2$
        2. Known T_1, convection boundary condition
- Knowing the temperature profile, we can determine the performance (e.g. heat flux/flow) and other temperature-related properties
- Consider a chip on a circuit board, with a measured power consumption (known $\dot{Q}$ and $\dot{q}$); we also know the thickness of the circuit board, so we have the heat transfer through the circuit board
    - In a real system this is often much more complicated, e.g. the circuit board can have multiple layers and vias, different materials, and a copper heat sink
- The *thermal resistance* approach is a convenient way to analyze complex systems
    - Directly analogous to electrical circuits
    - In a circuit we have flow = driving force divided by resistance
    - In heat transfer $\dot{Q} = \frac{T_1 - T_2}{R}$

Definition

The thermal resistance R is defined such that $\dot{Q} = \frac{T_1 - T_2}{R}$

- For conduction $\dot{Q} = \frac{kA}{L}(T_1 - T_2) \implies R = \frac{L}{kA}$, with units of K/W
- For convection $\dot{Q} = hA(T_1 - T_2) \implies R = \frac{1}{hA}$
- For radiation we have to take a shortcut: $\dot{Q} = \varepsilon\sigma A(T_s^4 - T_{surr}^4)$ is nonlinear, so we force it into

$h_{rad}A(T_s - T_{surr})$

  - $h_{rad} = \dfrac{\varepsilon\sigma A(T_s^4 - T_{surr}^4)}{A(T_s - T_{surr})} = \varepsilon\sigma(T_s^2 + T_{surr}^2)(T_s + T_{surr})$
  - This a directly a function of T_s and T_{surr}; since T_s is often unknown, we often take a guess and do a question, and then come back later to refine our guess if necessary

Summary

Thermal resistances for the different heat transfer mechanisms in steady state:

- Conduction:
$$R = \frac{L}{kA}$$
- Convection:
$$R = \frac{1}{hA}$$
- Radiation:
$$R = \frac{1}{h_{rad}A}$$
where
$$h_{rad} = \varepsilon\sigma(T_s^2 + T_{surr}^2)(T_s + T_{surr})$$

- If we have multiple layers of materials in series, we can consider it just like we would consider series resistances in a circuit
  - The equivalent heat resistance of multiple layers in series is just the sum of the heat resistances
  - We can combine all the $\frac{L}{k}$, $\frac{1}{h}$ and $\frac{1}{h_{rad}}$ into $\frac{1}{U}$ where U is the overall heat transfer coefficient, so $UA = \frac{1}{R_{tot}}$

Lecture 23, Nov 1, 2022

Thermal Resistances in Parallel

- Need to assume each side has the same uniform temperature across all different materials, and heat transfer is only 1D (no heat transfer between the different thermal resistances)
- The total heat flux is $\dot{Q} = \sum_i \dot{Q}_i = \sum_i \frac{T_1 - T_2}{R_i} = (T_1 - T_2)\left(\sum_i \frac{1}{R_i}\right)$
- This gives us $\frac{1}{R_{tot}} = \sum_i \frac{1}{R_i}$, completely analogous to electrical resistors

Thermal Resistance Networks

- We can combine complex heat transfer conditions into resistance networks
- Simplify using series and parallel resistance rules like in circuits
- Main assumptions:
  - 1D heat flow
  - Isothermal normal to heat flow

Thermal Contact Resistance

- So far we've assumed that at the boundary the temperatures are identical, but this assumes materials are completely flush against each other
- Real surfaces are rough (nano scale topology)
  - Roughness is measured in nanometers

- Due to the roughness the two surfaces are not in perfect contact, so at the boundary there is a slight temperature difference
  - Instead of having $T_1 \to T_2 \to T_3$ we actually have $T_1 \to T_2 \to T_2' \to T_3$
  - Define $\Delta T = T_2 - T_2'$
- Since the air between the layer is a poor conductor in reality most of the heat flow goes through the parts of the surface that are actually in contact
  - The effective heat transfer area is only the area in contact
  - To minimize thermal contact resistances, we can fill in the gaps with a conductive material, e.g. silicon oil, glycerol
    - This is how thermal paste works
- Define the *thermal contact resistance* $R_c = \dfrac{\Delta T}{\dot{q}}$ with units of $\mathrm{m^2K/W}$
  - Notice this is defined per unit of heat flux, not per unit of heat transfer
  - Same unit as $\dfrac{1}{h}$ but not $\dfrac{1}{hA}$ like the other resistances
- Define $h_c = \dfrac{1}{R_c} = \dfrac{\dot{q}}{\Delta T}$ as the *thermal contact conductance*
  - $\dot{q} = h_c \Delta T, \dot{Q} = h_c A \Delta T$

Lecture 24, Nov 3, 2022

Heat Conduction in Cylinders and Spheres

- Consider a cylinder with a z axis and a radial axis r, with pipe length L
  - For a long pipe $\dfrac{\mathrm{d}T}{\mathrm{d}z} \ll \dfrac{\mathrm{d}T}{\mathrm{d}c}$ so we can assume 1D heat conduction
- We want to solve the heat conduction equation to get $T(r)$
- Recall $\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r}\right) = 0$ at steady state, with boundary conditions $T(r_1) = T_1, T(r_2) = T_2$
  - Integrate: $r\dfrac{\mathrm{d}T}{\mathrm{d}r} = c_1 \implies \dfrac{\mathrm{d}T}{\mathrm{d}r} = \dfrac{c_1}{r}$
  - Integrate again: $T(r) = c_1 \ln r + c_2$
  - Apply boundary conditions:
    - $T_1 = c_1 \ln r_1 + c_2, T_2 = c_1 \ln r_2 + c_2$
    - Take the difference: $T_1 - T_2 = c_1 \ln \dfrac{r_1}{r_2}$
    - $c_1 = \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}}$
    - Plug this back in and we get $T_2 = \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} + c_2$ so $c_2 = T_2 - \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln r_2$
    - Plug back in and simplify: $T(r) = \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln(r) - \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln r_2 + T_2$
    - $T(r) = \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln \dfrac{r}{r_2} + T_2$
- Consider heat transfer, which is constant at steady state
  - $\dot{Q}_{cond} = -kA_1 \left.\dfrac{\mathrm{d}T}{\mathrm{d}r}\right|_{r=r_1} = -kA_2 \left.\dfrac{\mathrm{d}T}{\mathrm{d}r}\right|_{r=r_2}$
  - $\dfrac{\mathrm{d}T}{\mathrm{d}r} = \dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \dfrac{1}{r}$
  - $\dot{Q}_{cond} = -k(2\pi r_1 L)\left(\dfrac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \dfrac{1}{r_1}\right) = \dfrac{2\pi L k}{\ln \frac{r_2}{r_1}}(T_1 - T_2)$
- Define the thermal resistance of a cylinder as $R = \dfrac{T_1 - T_2}{\dot{Q}} = \dfrac{\ln \frac{r_2}{r_1}}{2\pi L k}$

- For a sphere, we can do a similar derivation and get $R = \frac{r_2 - r_1}{4\pi r_1 r_2 k}$

Summary

For 1D heat conduction in a cylinder:

$$T(r) = \frac{T_1 - T_2}{\ln\left(\frac{r_1}{r_2}\right)} \ln\left(\frac{r}{r_2}\right) + T_2$$

which gives a thermal resistance of

$$R_{\text{cylinder}} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi L k}$$

For a sphere this is

$$R_{\text{sphere}} = \frac{r_2 - r_1}{4\pi r_1 r_2 k}$$

For heat convection and radiation, the equations are the same as the Cartesian case, but note areas are no longer constant

Convection in Cylinders and Spheres

- Consider water in a pipe, with water $T_{\infty,1}$ and inner heat transfer coefficient h_1 and surrounding air $T_{\infty,2}$ and outer h_2
- For the total heat transfer we need to consider the convection from the water to the pipe, conduction through the pipe and convection to the outside air
- The main thing to watch out for is that the areas are not constant in cylindrical coordinates
- $R_{total} = \frac{1}{2\pi r_1 L h_1} + \frac{\ln\frac{r_2}{r_1}}{2\pi L k} + \frac{1}{2\pi r_2 L h_2}$

Lecture 25, Nov 14, 2022

Insulation

- The R-value of insulation indicates thermal resistance
  - Note this R value normalized is per unit area (temperature difference per heat flux)
  - $R = \frac{\Delta T}{\dot{q}} = \frac{L}{k}$, note area is not in here
  - Typically in imperial units in BTU per (hour foot degree Fahrenheit)
- Insulation is typically made of materials containing small air pockets, to reduce conduction and also convection

Critical Radius of Insulation

- Consider a pipe with insulation starting at r_1 and ending at r_2
- The insulation increases the wall thickness; typically increases conductive resistance $R_{cond} = \frac{L}{kA}$
- However convective heat resistance is *decreased* due to the increase in area, so insulation can increase heat transfer!
- $\dot{Q} = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{\ln\frac{r_1}{r_2}}{2\pi L k} + \frac{1}{2\pi r_2 L h}}$
  - There is a critical radius of insulation where $\dot{Q}$ is the maximum
  - The critical point is $r_{crit} = \frac{k}{h}$; below this point, insulation increases the heat transfer; above this point insulation decreases it

Heat Flow Through Finned Surfaces

- Assumption: 1D conduction (i.e. temperature is uniform in the y and z directions)
- Fin with cross-sectional area A_c, length L, transferring heat into fluid with T_∞
- Consider an infinitesimal slice of the fin; $\dot{Q}_x$ would not be constant due to heat loss through the sides of the fin
  - Energy balance: $\dot{Q}_{cond,x} = \dot{Q}_{cond,x+\Delta x} + \dot{Q}_{conv}$
  - Let P be the perimeter of the fin, so the surface area is $P\Delta x$, so $\dot{Q}_{conv} = hP\Delta x(T(x) - T_\infty)$
  - $\dot{Q}_{cond,x+\Delta x} - \dot{Q}_{cond,x} + hP\Delta x(T(x) - T_\infty) = 0$
  - Take $\lim\limits_{\Delta x \to 0}$ and we get $\dfrac{\mathrm{d}Q_{cond}}{\mathrm{d}x} + hP(T(x) - T_\infty) = 0$
  - We know $\dot{Q}_{cond} = -kA_c\dfrac{\mathrm{d}T}{\mathrm{d}x}$ so $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(-kA_c\dfrac{\mathrm{d}T}{\mathrm{d}x}\right) + hP(T(x) - T_\infty) = 0$
  - We usually assume A_c, P, k are constant, so $\dfrac{\partial^2 T}{\partial x^2} - \dfrac{hP}{kA_c}(T(x) - T_\infty) = 0$
- Let $\theta = T - T_\infty$ and $a^2 = \dfrac{hP}{kA_c}$
  - Note $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{\mathrm{d}T}{\mathrm{d}x}$
  - $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - a^2\theta = 0$
  - $\theta(x) = c_1e^{ax} + c_2e^{-ax}$
- Using boundary conditions: $T(0) = T_b$
  - If we assume a fin that's infinitely long, $T(L) = T_\infty \implies \theta(0) = \theta_b, \theta(L) = 0$
    - $\theta(L) = c_1e^{ax} = 0 \implies c_1 = 0$
    - $\theta(0) = c_2 = \theta_b$
    - $\theta(x) = \theta_b e^{-ax}$ or $\dfrac{\theta(x)}{\theta_b} = e^{-ax}$ or $\dfrac{T(x) - T_\infty}{T_b - T_\infty} = \exp\left(-x\sqrt{\dfrac{hP}{kA_c}}\right)$
    - $\left.\dfrac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=0} = \left.\dfrac{\mathrm{d}Q}{\mathrm{d}x}\right|_{x=0} = -\theta_b a$
    - For an infinitely long fin $\dot{Q} = \sqrt{hPkA_c}(T_b - T_\infty)$

Summary

For an infinitely long fin, the temperature profile varies as:

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-ax}$$

where

$$a = \sqrt{\frac{hP}{kA_c}}$$

where P is the perimeter of the fin, A_c is the cross-sectional area of the fin, and the total heat transfer *through the fin* is

$$\dot{Q} = \sqrt{hPkA_c}(T_b - T_\infty)$$

Lecture 26, Nov 15, 2022

More On Finned Surfaces

- Why does longer hair not lead to more heat transfer?
  - k for hair is very low, so longer hair leads to a negligible amount of additional heat transfer
  - The additional hair creates a boundary layer that effectively lowers h leading to less convection

- Consider cylindrical fins, increasing the diameter from d to $2d$ results in $\frac{\dot{Q}'}{\dot{Q}} = 2^{\frac{3}{2}}$, an increase in the total heat transfer
  - For heat flux however we get $\frac{\dot{q}'}{\dot{q}} = 2^{-\frac{1}{2}}$, which is lower
  - When designing a heat sink it might be better to have a larger number of smaller fins
    - Note: Having fins that are too small might break up the flow and change h, leading to worse performance

Finite Length Fins

- Method 1: Consider an adiabatic tip (insulated tip)
  - No heat transfer at the tip means the temperature at the tip must be constant
  - $\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=L} = \left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{x=L} = 0$
  - $\theta(x) = c_1 e^{ax} + c_2 e^{-ax} \implies \frac{\mathrm{d}\theta}{\mathrm{d}x} = c_1 a e^{ax} - c_2 a e^{-ax}$
  - Plug in the initial condition, $0 = c_1 a e^{aL} - c_1 a e^{-aL}$
  - At $x = 0, \theta = \theta_b$, so $\theta_b = c_1 + c_2$
  - Solving yields $\frac{\theta(x)}{\theta_b} = \frac{\cosh(a(L-x))}{\cosh(aL)}$
  - Solving for $\dot{Q}$ using Fourier's law yields $\sqrt{hPkA_c}(T_s - T_\infty)\tanh(aL)$
    - Note $\tanh(L) \to 1$ as $L \to \infty$, so this approaches the infinitely long fin equation as the fin gets longer
- Method 2: Use the "corrected length"
  - Have the convection coming out of the tip be idealized as coming out from the fin side
  - Imagine extending the fin by ΔL such that the additional side area ΔLP equals the size of the tip cross-section, now we can assume the tip is adiabatic
  - Corrected length is $L_c = L + \frac{A_c}{P}$
  - Use this L_c with the adiabatic tip solution for T, θ
  - Note this only works well if $A_c \ll L$ so the extension is minimal

Summary

For a fin with finite length L and an insulated tip, then

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cos(a(L-x))}{\cosh(aL)}$$

where

$$a = \sqrt{\frac{hP}{kA_c}}$$

and the total heat transfer through the fin is

$$\dot{Q} = \sqrt{hPkA_c}(T_s - T_\infty)\tanh(aL)$$

For fins with cross sectional area small relative to the length and non-adiabatic tip, use the corrected length $L_c = L + \frac{A_c}{P}$

Fin Efficiency

- The most "efficient" fin would have effectively infinite conductivity, so $T(x) = T_b$ for all x so that the convection along the fin is maximized

- In this case the heat transfer is just convection at uniform temperature, $\dot{Q}_{max} = hA_{fin}(T_b - T_\infty) = hPL(T_b - T_\infty)$ ignoring the fin tip
- Define the fin efficiency as $\eta_{fin} = \dfrac{\dot{Q}}{\dot{Q}_{max}}$
  - This is equal to $\dfrac{\sqrt{hPkA_c}(T_b - T_\infty)}{hPL(T_b - T_\infty)} = \dfrac{1}{aL}$ for an infinitely long fin
  - For an adiabatic tip $\dfrac{\sqrt{hPkA_c}(T_b - T_\infty)\tanh(aL)}{hPL(T_b - T_\infty)} = \dfrac{\tanh(aL)}{aL}$
  - As L increases, η_{fin} approaches 0
- The fin effectiveness is defined as $\varepsilon_{fin} = \dfrac{\dot{Q}_{fin}}{\dot{Q}_0}$ where $\dot{Q}_0$ is the heat transfer without the fin, which would be $hA_c(T_b - T_\infty)$
  - For an infinitely long fin $\varepsilon_{fin} = \sqrt{\dfrac{kP}{hA_c}}$
- To increase the fin effectiveness, maximize k and $\dfrac{P}{A_c}$
- When h goes up, the fin effectiveness goes down; fins are the most effective with low h
  - e.g. if we have a boundary between air and water, it's better to have the fin on the air side since h in air is much lower
- Rule of thumb: Fins are worth it if $\varepsilon \geq 2$

Lecture 27, Nov 17, 2022

Fin Sizing

- When can we assume a fin to be infinitely long?
- Compare adiabatic tip vs. infinitely long fin equation, taking the ratio we get $\tanh(aL)$
- For $aL = 1$, $\tanh(aL) = 0.762$; for $aL = 5$, $\tanh(aL) = 0.9999$)
  - As a rule of thumb if $aL \geq 5$ we can assume the fin is infinitely long
  - A value of 1 gives 76.2% of the total heat transfer from an infinitely long fin, with a lot less material
- $L = \dfrac{1}{a}$ is typically a reasonable length for a fin
- What about the area of fins?
  - Consider area with fin and area without fin
  - $\dot{Q}_{total} = \dot{Q}_{nofin} + \dot{Q}_{fin}$
  - From the fin efficiency definition $\eta_{fin} = \dfrac{\dot{Q}_{fin}}{hA_{fin}(T_b - T_\infty)}$
  - $\dot{Q}_{fin} = h\eta_{fin}A_{fin}(T_b - T_\infty)$
  - Putting it all together $\dot{Q}_{total} = h(A_{nofin} + \eta_{fin}A_{fin})(T_b - T_\infty)$
- $(A_{nofin} + \eta_{fin}A_{fin})$ is the *effective area for heat transfer*
  - In terms of thermal resistances $R = \dfrac{T_b - T_\infty}{\dot{Q}_{total}} = \dfrac{1}{h(A_{nofin} + \eta_{fin}A_{fin})}$

Transient Heat Conduction (Lumped)

- Simple example: taking a material and immersing it in a fluid with a high temperature difference, resulting in rapid heat transfer
  - What is $T(t)$?
- Recall $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$
- Lumped capacitance: simple assumption that there is no temperature gradient in the body, i.e. $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{\partial^2 T}{\partial y^2} = \dfrac{\partial^2 T}{\partial z^2} = 0$, the temperature is at uniform temperature everywhere

  - In reality modes of heat transfer include conduction within the material and convection to the surrounding fluid
  - This makes sense only if $R_{cond} \ll R_{conv}$
- Energy balance: Let $\Delta \dot{E}(t) = -\dot{Q}_{conv}(t)$ be the energy change in the body
  - $\Delta \dot{E}(t) = mc_p \frac{\mathrm{d}T}{\mathrm{d}t} = \rho V c_p \frac{\mathrm{d}T}{\mathrm{d}t}$
  - $hA(T - T_\infty) = -\rho V c_p \frac{\mathrm{d}T}{\mathrm{d}t} \implies \frac{\mathrm{d}T}{T - T_\infty} = \frac{\mathrm{d}(T - T\infty)}{T - T_\infty} = -\frac{hA}{\rho V c_p} \mathrm{d}t$
  - Integrate to get $\ln(T - T_\infty) = -\frac{hA}{\rho V c_p} t + c_1$, apply boundary condition that $T(0) = T_i$
  - $\ln\left(\frac{T - T_\infty}{T_i - T_\infty}\right) = -\frac{hA}{\rho V c_p}$
  - This compares the "changing driving force" against the "max driving force"
- Define the time constant $\tau = \frac{\rho V c_p}{hA}$ so $\frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{t}{\tau}}$

Equation

For lumped transient heat conduction,

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{t}{\tau}}$$

where $\tau = \frac{\rho V c_p}{hA}$

Validity of the Lumped Capacitance Assumption

- This only makes sense if $R_{conv} \gg R_{cond}$
- Take $\frac{R_{cond}}{R_{conv}} = \frac{\frac{L}{kA}}{\frac{1}{hA}} = \frac{hL}{k}$
- $\mathrm{Bi} = \frac{hL_c}{k}$ is the *Biot number*, a dimensionless quantity
  - L_c is a characteristic length in the direction of conduction, from the midpoint to the wall
  - $L_c = \frac{V}{A}$
  - For a sphere $L_c = \frac{r}{3}$, for a cylinder (with length $\gg$ radius) $L_c = \frac{r}{2}$
- Consider the steady state analogue; for $\mathrm{Bi} \gg 1$, the temperature drops the sharpest over conduction, so lumped capacitance is not valid; for $\mathrm{Bi} \ll 1$, the temperature drops the sharpest over convection, so lumped capacitance is valid
- Typical cutoff is $\mathrm{Bi} < 0.1$
- Example: putting steel rod at 300°C into furnace at 1200°C with $h = 100\mathrm{W/m^2\,K}, D = 0.1m, k = 51.2\mathrm{W/m\,K}, \rho = 7832\mathrm{kg}m^3, c = 541\mathrm{J/kg\,K}$, how long until the rod temperature reaches 800°C?
  - First, check validity of lumped capacitance assumption: $\mathrm{Bi} = \frac{hL_c}{k} = \frac{100\mathrm{W/m^2k} \cdot \frac{0.05\mathrm{m}}{2}}{51.2\mathrm{W/m\,K}} = 0.05 < 0.1$ so the assumption is valid
  - Using lumped capacitance $\frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{hA}{\rho V c} t} = e^{-\frac{h}{\rho c} \cdot \frac{2}{r} t}$
  - $\ln \frac{800 - 1200}{300 - 1200} = \frac{-2 \cdot 100\mathrm{W}m^2.K}{7832\mathrm{kg/m^3} \cdot 541\mathrm{J}kg.K} t$
  - $t = 859\mathrm{s}$

Lecture 28, Nov 21, 2022

Transient Conduction (Non-Lumped)

- When $B_i > 0.1$ we can no longer neglect the temperature difference in the body, so we have to solve the complete heat conduction equation
- When we insert a plane wall into a fluid, we should see a bump in the temperature in the middle of the wall that becomes flatter over time
- In this case $T = T(x, t)$ if we assume 1D conduction
- $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$
  - $\alpha = \dfrac{k}{\rho c_p}$ is the thermal diffusivity
- Initial conditions: $T(x, 0) = T_i$, boundary conditions: at L, $-k\dfrac{\partial T}{\partial x} = h(T(L, t) - T_\infty)$
  - Use symmetry, $\left.\dfrac{\partial T}{\partial x}\right|_{x=0} = 0$ (note zero is the centre of the plate)
- Use a change of variables $X = \dfrac{x}{L}, \mathrm{Bi} = \dfrac{hL}{k}, \tau = \dfrac{\alpha t}{L^2}$
  - $\tau = \mathrm{Fo}$ is also known as the Fourier number, a unitless measure of time (note this is a variable)
    - * The Fourier number is the ratio of conductive heat transfer to energy increase in the system
    - * Imagine a cube with sides L with conductive heat transfer in, convective heat transfer out
    - * From Fourier's law $\dot{Q}_{cond} = kA\dfrac{\mathrm{d}T}{\mathrm{d}x} = kL^2\dfrac{\Delta T}{L} = kL\Delta T$
    - * $\Delta E = mc\dfrac{\mathrm{d}T}{\mathrm{d}t} = \rho L^3 c\dfrac{\Delta T}{t}$
    - * $\dfrac{\dot{Q}_{cond}}{\Delta E} = \dfrac{k}{\rho c}\dfrac{t}{L^2} = \dfrac{\alpha t}{L^2} = \mathrm{Fo}$
  - $\theta(X, \mathrm{Fo}) = \dfrac{T - T_\infty}{T_i - T_\infty}$ is the normalized thermal driving force
- $\dfrac{\partial^2 \theta}{\partial X^2} = \dfrac{\partial \theta}{\partial \mathrm{Fo}}$
  - Boundary conditions $\left.\dfrac{\partial \theta}{\partial X}\right|_{X=0} = 0$
  - At the walls $\left.\dfrac{\partial \theta}{\partial X}\right|_{X=1} = -\mathrm{Bi}\theta(1, \mathrm{Fo})$
- This can be solved analytically, but we're not going to do so
- In the simplest case of the lumped capacitance $\theta = e^{-\frac{hA}{\rho V c}t} = e^{-\left(\frac{hL}{k}\right)\left(\frac{\alpha t}{L^2}\right)t} = e^{-\mathrm{BiFo}}$
- In more complex cases $\theta = \theta(X, \mathrm{Bi}, \mathrm{Fo})$ is a complicated function, but the textbook gives some approximations for different geometries for sufficiently large t
  - For the plane wall: $\theta = \dfrac{T(x, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \mathrm{Fo}} \cos\left(\lambda_1 \dfrac{x}{L}\right)$
    - * A_1, λ_1 are functions of Bi
    - * $\theta_0 = A_1 e^{-\lambda_1^2 \mathrm{Fo}}$
    - * $\theta = \theta_0(\mathrm{Fo}) \cos\left(\lambda_1 \dfrac{x}{L}\right)$
  - For all the geometries, we can separate θ into a function of time and a function of position

Summary

For non-lumped transient conduction for a plane wall:

$$\theta(x,t) = A_1 e^{-\lambda_1^2 \mathrm{Fo}} \cos\left(\lambda_1 \frac{x}{L}\right)$$

where $\mathrm{Fo} = \frac{\alpha t}{L^2}$ is the Fourier number, $\alpha = \frac{k}{\rho c}$ is the thermal diffusivity, and A_1, λ_1 are functions of $\mathrm{Bi} = \frac{hL}{k}$, the Biot number, which can be determined through a table

Lecture 29, Nov 22, 2022

Semi-Infinite Solids

- Consider an object with surface temperature T_s and internal temperature T_s
  - The skin layer is the outer layer of the solid where the temperature is a gradient; heat transfer is meaningfully occurring
  - The core is the part that's relatively untouched by heat transfer so it has a roughly constant temperature
  - The actual temperature distribution would be an exponential, and the skin layer is the region where the exponential is changing fast, whereas the core is the asymptote
  - The dividing point is relatively subjective
- How does the skin depth δ vary with time?
- Apply a semi-quantitative scaling analysis, with the goal of finding functional relationships
  - Starting with the conduction equation: $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
  - $\frac{\partial^2 T}{\partial x^2} \sim \frac{\left.\frac{\partial T}{\partial x}\right|_{x=\delta} - \left.\frac{\partial T}{\partial x}\right|_{x=0}}{\delta}$
  - $\left.\frac{\partial T}{\partial x}\right|_{x=\delta} = 0$ since at that point the heat transfer is done, so temperature is not changing much
  - $\left.\frac{\partial T}{\partial x}\right|_{x=0} \approx \frac{T_i - T_s}{\delta}$ is the slope roughly at the surface
  - $\frac{\partial^2 T}{\partial x^2} \sim \frac{0 - \frac{T_i - T_s}{\delta}}{\delta} = \frac{T_s - T_i}{\delta^2}$
  - $\frac{\partial T}{\partial t} \sim \frac{T_s - T_i}{\Delta t} = \frac{T_s - T_i}{t}$
  - Substituting: $\frac{T_s - T_i}{\delta^2} \approx \frac{1}{\alpha}\frac{T_s - T_i}{t} \implies \delta(t) \sim \sqrt{\alpha t}$
- δ scales with $\sqrt{\alpha t}$
  - This is not an exact equivalence, but now we know roughly how deep the heat transfer gets as time goes on
- Consider a sphere with radius r_0, then heat transfer reaches the centre when $\delta = r_0$; so we can devise a characteristic time $t_c = \frac{r_0^2}{\alpha}$ for the heat transfer to reach the centre
  - If $t \ll t_c$ then we can treat the body as *semi-infinite*, i.e. infinite in one direction
- With a semi-infinite assumption we have an exact solution to the transient heat conduction problem
  - $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ with boundary conditions $T(0,t) = T_s, T(\infty, t) = T_i, T(x,0) = T_i$
  - Using the scaling analysis to relate t and x
    - $\delta(t) \sim \sqrt{\alpha t}$
    - Define the similarity variable $\eta = \frac{x}{2\delta} = \frac{x}{2\sqrt{\alpha t}}$

    - $\frac{\partial T}{\partial t} = \frac{\mathrm{d}T}{\mathrm{d}\eta}\frac{\partial \eta}{\partial t} = \frac{\mathrm{d}T}{\mathrm{d}\eta}\left(\frac{-x}{4t\sqrt{\alpha t}}\right)$
    - $\frac{\partial T}{\partial x} = \frac{\mathrm{d}T}{\mathrm{d}\eta}\frac{\partial \eta}{\partial x} = \frac{\mathrm{d}T}{\mathrm{d}\eta}\left(\frac{1}{2\sqrt{\alpha t}}\right)$
    - $\frac{\partial^2 T}{\partial x^2} = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\mathrm{d}T}{\mathrm{d}\eta}\right)\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{\mathrm{d}^2 T}{\mathrm{d}x^2}\left(\frac{1}{4\alpha t}\right)$
    - $\frac{1}{4\alpha t}\frac{\mathrm{d}^2 T}{\mathrm{d}\eta^2} = \frac{1}{\alpha}\left(-\frac{x}{4t\sqrt{\alpha t}}\right)\frac{\mathrm{d}T}{\mathrm{d}\eta}$
    - $\frac{\mathrm{d}^2 T}{\mathrm{d}\eta^2} = -\frac{x}{\sqrt{\alpha t}}\frac{\mathrm{d}T}{\mathrm{d}\eta} = -2\eta\frac{\mathrm{d}T}{\mathrm{d}\eta}$
  - New boundary conditions: $T(0) = T_s, T(\infty) = T_i$
  - Let $w = \frac{\mathrm{d}T}{\mathrm{d}\eta} \implies \frac{\mathrm{d}w}{\mathrm{d}\eta} = -2\eta w$
  - Solve: $\ln w = -\eta^2 + C \implies w = c_0 e^{-\eta^2} = \frac{\mathrm{d}T}{\mathrm{d}\eta}$
  - $T = c_0 \int_0^\eta e^{-u^2}\,\mathrm{d}u + c_1$
  - Boundary conditions: $T(0) = c_1 = T_s, T(\infty) = c_0\frac{\sqrt{\pi}}{2} + T_s \implies c_0 = \frac{2(T_i - T_s)}{\sqrt{\pi}}$
  - $\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}\,\mathrm{d}u = \mathrm{erf}(\eta)$
  - $1 - \frac{T - T_s}{T_i - T_s} = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}(\eta)$
    - erfc is the complementary error function
  - $\frac{T - T_i}{T_s - T_i} = \mathrm{erfc}(\eta)$
- For heat flux: $\dot{q}_s = -h\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=0} = -h\left.\frac{\mathrm{d}T}{\mathrm{d}\eta}\frac{\partial \eta}{\partial x}\right|_{\eta=0}$
  - Differentiating the temperature profile we get $\frac{1}{T_i - T_s}\frac{\mathrm{d}T}{\mathrm{d}\eta} = \frac{2}{\sqrt{\pi}}e^{-\eta^2} \implies \frac{\mathrm{d}T}{\mathrm{d}\eta} = \frac{2(T_i - T_s)}{\sqrt{\pi}}e^{-\eta^2}$
  - $\frac{\partial \eta}{\partial x} = \frac{1}{2\sqrt{\alpha t}}$
  - Plugging these in $\dot{q} = -k(T_i - T_s)\cdot\frac{2}{\sqrt{\pi}}\cdot\frac{1}{2\sqrt{\alpha t}}$
  - Simplify to get a heat flux at the base of $\dot{q} = \frac{k(T_s - T_i)}{\sqrt{\pi\alpha t}}$
- Contact of two semi-infinite bodies: joining together two semi-infinite materials A and B, applying the same analysis as before
  - We have $T_{s,A} = T_{s,B} = T_s$ and $\dot{q}_{s,A} = \dot{q}_{s,B}$
  - $-\frac{k(T_s - T_{A,i})}{\sqrt{\pi\alpha_A t}} = \frac{k_B(T_s - T_{B,i})}{\sqrt{\pi\alpha_B t}}$
  - $\frac{T_{A,i} - T_s}{T_s - T_{B,i}} = \frac{\sqrt{(k\rho c)_B}}{\sqrt{(k\rho c)_A}} = \frac{\gamma_B}{\gamma_A}$
    - γs are known as the *effusivities*
  - $T_s = \frac{\gamma_A T_{A,i} + \gamma_B T_{B,i}}{\gamma_A + \gamma B}$
    - Notice this is constant

Summary

When the time scale is such that the skin depth $\delta = \sqrt{\alpha t} \ll L$ where L is the characteristic length, we can treat a solid as semi-infinite, in which case

$$\frac{T - T_i}{T_s - T_i} = \operatorname{erfc}(\eta) = 1 - \operatorname{erf}(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2}\,\mathrm{d}u$$

where $\eta = \dfrac{x}{2\sqrt{\alpha t}}$ is the similarity variable; this results in a heat transfer at the base of

$$\dot{q} = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

This usually applies in cases of very low Bi, i.e. $R_{cond} \gg R_{conv}$

Lecture 30, Nov 24, 2022

Forced Convection

- Force convection is convection in which the fluid is imparted by external means, as opposed to natural convection which relies on natural fluid motion caused by density changes from temperature
- Up until now we've been using $\dot{Q} = hA(T_i - T_\infty)$ with h already given
    - h could be measured experimentally in a complex system
    - Can we determine h based on system properties?
- Relevant properties:
    - Viscosity μ
    - Density ρ
    - Thermal conductivity k
    - Heat capacity c_p
    - Fluid velocity v_∞ (for a free stream, outside the boundary layer)
    - Shape and size:
        - Characteristic length (length of a plate, diameter of cylinder/sphere)
    - Type of flow (laminar vs turbulent)
- We need to consider the boundary layer, where the fluid slows down near the plate due to the no-slip condition at the boundary
    - We consider this as the region where $v < 0.99v_\infty$
- At the interface between fluid and solid, heat transfer occurs only by conduction
    - $\dot{q} = -k\left.\dfrac{\partial T}{\partial y}\right|_{y=0}$
    - $\dot{q} = h(T_s - T_\infty)$ by Newton's law of cooling
    - Therefore $h(T_s - T_\infty) = -k\left.\dfrac{\partial T}{\partial y}\right|_{y=0} \implies h = \dfrac{-k\left.\frac{\partial T}{\partial y}\right|_{y=0}}{T_s - T_\infty}$
    - Now the problem becomes solving for $\dfrac{\partial T}{\partial y}$, but typically we don't know this
- Consider the boundary layer and assume $T_s > T_\infty$
    - We have both a velocity and a thermal boundary layer; we basically want the slope of this thermal boundary layer
    - Overall h changes with position as local $\dfrac{\partial T}{\partial y}$ at the surface changes, as the boundary layer develops (boundary layer gets thicker as the fluid flows further down the surface)
    - Define a local heat transfer coefficient x where x is along the surface, then the overall h is

$$\bar{h} = \frac{1}{L}\int_0^L h(x)\,\mathrm{d}x$$

Boundary Layer Flow

- Typically the boundary layer begins with laminar flow, then becomes turbulent as you go down the surface, with a transitional region in the middle
- The fluid exerts a stress on the plate, $\tau = \mu \left.\frac{\partial v}{\partial y}\right|_{y=0}$
- We need to define a "friction coefficient" for the fluid
  - By conservation of energy $\Delta(p\dot{v}) + \Delta\left(\frac{\dot{m}v^2}{2}\right) = 0$
  - $p - p_\infty + \frac{\rho v^2}{2} - \frac{\rho v_\infty^2}{2} = 0 \implies p - p_\infty = \frac{\rho v_\infty^2}{2}$
    - This $p - p_\infty$ is the force per unit area felt by the body
    - $\frac{\rho v_\infty^2}{2}$ is the inertial force
  - $\frac{F}{A} = \tau = c_F\frac{\rho v_\infty^2}{2}$ where c_F is the friction coefficient for the fluid, defined for different shapes, generally about 1
    - For a sphere it's about 0.47, for a plate about 1.17, for a convex hull 2.3, for an airfoil 0.04

Diffusivities

- In molecular diffusion (mass transfer) we have Fick's law $J_A = -D_{AB}\frac{\mathrm{d}C_A}{\mathrm{d}x}$
  - D_{AB} is the mutual diffusion constant for A into solid B with units of m^2/s
  - C_A is the concentration of A
- In heat transfer we have Fourier's law $\dot{q} = -k\frac{\mathrm{d}T}{\mathrm{d}x} \implies \dot{q} = \frac{k}{\rho c_p}\frac{\mathrm{d}}{\mathrm{d}x}(\rho c_p T) = -\alpha\frac{\mathrm{d}}{\mathrm{d}x}(\rho c_p T)$
  - α is the thermal diffusivity
  - $\rho c_p T$ can be thought of as a "concentration of thermal energy", the amount of thermal energy per unit volume with units of $\mathrm{J/m}^3$
- In a fluid μ is the dynamic viscosity; define $\frac{\mu}{\rho} = \nu$ to be the kinematic viscosity (aka momentum diffusivity)
  - Shear stress is in general $\tau = \mu\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{\mu}{\rho}\frac{\mathrm{d}}{\mathrm{d}y}(\rho v) = \nu\frac{\mathrm{d}}{\mathrm{d}y}(\rho v)$
  - This is the diffusion equation again
  - τ can be thought of as a "momentum flux"
  - ρv is the "concentration of momentum", momentum per unit volume

Lecture 31, Nov 28, 2022

Nusselt Number

- In a turbulent flow we typically have greater heat transfer and shear stress
- The transition point from laminar to turbulent depends on the Reynolds number, the ratio of inertia to viscosity in the fluid
  - For us the characteristic length used is the x position along the plate
- For every geometry, there is a critical Reynolds number at which the transition happens
  - For a flat plate this is about 5×10^5
- We have 2 boundary layers, the velocity boundary layer and the temperature boundary layer; the size of one may be smaller or larger than the other, depending on fluid properties
  - Fluids with high kinematic viscosity (momentum diffusivity, e.g. oils) have thick velocity boundary layers

    - Fluids with high thermal diffusivity ($\alpha = \frac{k}{\rho c}$) have thick thermal boundary layers
- The ratio of the boundary layers is described by the ratio of diffusivities $\frac{\nu}{\alpha}$
- Define the Prandtl number $\text{Pr} = \frac{\nu}{\alpha} = \frac{\frac{\mu}{\rho}}{\frac{k}{\rho c}} = \frac{\mu c}{k}$
    - For $\text{Pr} \ll 1$ (e.g. liquid metals), the thermal boundary layer is larger than the velocity boundary layer
    - For $\text{Pr} \gg 1$ (e.g. oils), the velocity boundary layer is larger than the thermal boundary layer
    - For $\text{Pr} \approx 1$ (e.g. gases), the boundary layers are comparable in size
- We can non-dimensionalize h
    - Convective heat transfer scales with D^2
    - If the fluid is not moving, we just have conduction, which scales with kD
    - How much is heat transfer enhanced by the fluid motion?
    - Taking the ratio of these we get $\frac{hD}{k}$
- Define the Nousselt number $\text{Nu} = \frac{hL_c}{k}$ where L_c is a characteristic length, and k is thermal conductivity of the fluid
    - Looks similar to the Biot number, but the thermal conductivity here is of the fluid
- $\text{Nu} = f(\text{Re}, \text{Pr})$ and geometry, and this relationship can be determined experimentally
    - Typically $\text{Nu} = C_0\text{Re}^m\text{Pr}^n$, with C_0, m, n determined for different geometries

Summary

Typically the convective heat transfer coefficient can be found by

$$\text{Nu} = C_0\text{Re}^m\text{Pr}^n$$

where $\text{Nu} = \frac{hL_c}{k}$, $\text{Re} = \frac{\rho v L_c}{\mu} = \frac{\nu L_c}{\mu}$ and $\text{Pr} = \frac{\nu}{\alpha} = \frac{\mu c}{k}$ with C_0, m, n determined for different geometries

Lecture 32, Nov 29, 2022

Forced Convection

- Consider a plate with a boundary layer forming, with laminar, transitional, and turbulent zones, with height $\delta_v(x)$
    - The boundary layer becomes turbulent as shear stress goes down due to the velocity gradient decreasing as the boundary layer gets thicker
- Recall $\tau = C_F\frac{\rho v_\infty^2}{2}$
    - C_F, the friction coefficient, has a local value depending on where on the shape you are
    - In practice friction goes down in the laminar regime, goes up in the transition, and then goes down again the turbulent regime
        - * This is again due to the growth of the boundary layer
    - For laminar flow $C_{F,x} = \frac{0.664}{\text{Re}_x^{\frac{1}{2}}}$
        - * We can average this over the length of the plate: $C_F = \frac{1}{L}\int_0^L \frac{0.664}{\text{Re}_x^{\frac{1}{2}}}\,\mathrm{d}x = \frac{1.328}{\text{Re}_L^{\frac{1}{2}}}$
    - For turbulent flow ($\text{Re}_x > 5 \times 10^5$): $C_{F,x} = \frac{0.0592}{\text{Re}_x^{\frac{1}{2}}}$

    - This gives an average $C_F = \dfrac{0.074}{\mathrm{Re}_L^{\frac{1}{5}}}$
- Now consider the thermal boundary layer
  - $h = -\dfrac{k}{T_s - T_\infty} \dfrac{\mathrm{d}T}{\mathrm{d}y}\Big|_{y=0}$
  - $\dfrac{\mathrm{d}T}{\mathrm{d}y} \sim \dfrac{T_s - T_\infty}{\delta_T}$
  - As the thermal boundary layer thickness goes up, the convective heat transfer goes down
  - This gives a local Nousselt number $\mathrm{Nu}_x = \dfrac{h_x x}{k}$
    - For laminar flow: $\mathrm{Nu}_x = 0.332\mathrm{Re}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}$ for $\mathrm{Pr} \geq 0.6$
      - This gives an average of $\mathrm{Nu} = 0.664\mathrm{Re}_L^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}$
    - For turbulent flow: $\mathrm{Nu}_x = 0.0296\mathrm{Re}_x^{\frac{4}{5}}\mathrm{Pr}^{\frac{1}{3}}$ for $0.6 \leq \mathrm{Pr} \leq 60, 5 \times 10^5 \leq \mathrm{Re}_x \leq 1 \times 10^7$
      - This gives an average $\mathrm{Nu} = 0.037\mathrm{Re}_L^{\frac{4}{5}}\mathrm{Pr}^{\frac{1}{3}}$
- The film temperature is defined as the average of the surface and free temperatures $T_F = \dfrac{T_s + T_\infty}{2}$
  - This is to find the average fluid properties

Flows Over Cylinders and Spheres

- Behind a cylinder/sphere there is a turbulent wake, making the boundary layer behave erratically (flow separation, technically not turbulence)
  - Turbulence can still occur, if $\mathrm{Re} = \dfrac{vD}{\nu} > 2 \times 10^5$
- For a cylinder $\mathrm{Nu} = C\mathrm{Re}^m\mathrm{Pr}^n$ just like for a plate
- The Churchill and Bernstein correlation is more accurate and applies over a broader range
  - $\mathrm{Nu} = 0.3 + \dfrac{0.62\mathrm{Re}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}}{\left(1 + \left(\frac{0.4}{\mathrm{Pr}}\right)^{\frac{2}{3}}\right)^{\frac{1}{4}}}\left(1 + \left(\dfrac{\mathrm{Re}}{28200}\right)^{\frac{5}{8}}\right)^{\frac{4}{5}}$, applicable over $\mathrm{RePr} > 0.2$
- Flow over a sphere is similar: $\mathrm{Nu} = 2 + \left(0.4\mathrm{Re}^{\frac{1}{2}} + 0.06\mathrm{Re}^{\frac{2}{3}}\right)\mathrm{Pr}^{0.4}\left(\dfrac{\mu_\infty}{\mu_s}\right)^{\frac{1}{4}}$, valid for $3.5 \leq \mathrm{Re} \leq 80000, 0.7 \leq \mathrm{Pr} \leq 380$
  - Note all properties are evaluated at T_∞, not T_p like the others (except μ_s)

Summary

For laminar flow over a flat plate (average):

$$\mathrm{Nu} = 0.664\mathrm{Re}_L^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}, \mathrm{Pr} \geq 0.6$$

For turbulent flow over a flat plate (average):

$$\mathrm{Nu} = 0.037\mathrm{Re}_L^{\frac{4}{5}}\mathrm{Pr}^{\frac{1}{3}}, 0.6 \leq \mathrm{Pr} \leq 60, 5 \times 10^5 \leq \mathrm{Re} \leq 1 \times 10^7$$

where Re is the Reynolds number evaluated for the entire plate; material properties and Pr can be determined through the film temperature

$$T_F = \frac{T_s + T_\infty}{2}$$

Lecture 33, Dec 1, 2022

Thermal Radiation

- Consider an enclosure at temperature T_{surr} filled with vacuum, containing an object of temperature T_s
  - $T_s = T_{surr}$ at equilibrium
- Thermal radiation is energy emitted by matter as a result of its finite temperature
  - Radiation has wave patterns (EM waves)
  - Thermal radiation typically has wavelengths of 0.1 to 100µm
    - UV is 0.1 to 0.4µm
    - Visible radiation is 0.4 to 0.7µm
    - IR radiation is 0.7 to 100µm
- Radiation is released with energy level changes from an excited state
  - When radiation is absorbed we go from lower to higher energy states, similar to spectroscopy
  - For infrared radiation this corresponds to vibrational energy levels
  - Visible radiation corresponds to electronic energy levels (typically outer electrons)
- Radiation is a volumetric phenomenon, but most solids are "opaque", so emissions from within the object will just be immediately reabsorbed
  - This is why we usually consider it a surface property
- A blackbody is a perfect emitter and absorber of radiation
  - At a given temperature, no surface can emit more energy than a blackbody
  - It also emits radiation equally in all directions ("diffuse" emission)
- Stefan-Boltzmann Law: The radiation energy emitted by a blackbody per unit time per unit area is given by $E_b = \sigma T^4$ for the Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\,\mathrm{W/m^2\,K^4}$
  - E_b is the blackbody emissive power
- A blackbody is a theoretical object, but some things come close:
  - Black paint
  - Isothermal cavity (e.g. a box with a very small hole, the hole is a blackbody)
- The spectral distribution of blackbody radiation has the form $\dfrac{c_1}{\lambda^5 \exp\left(\frac{c_2}{\lambda T}\right) - 1}$
- In a real body, the radiation emission and absorption are dependent on wavelength and direction
  - e.g. CO_2's absorption spectrum absorbs more in the region of sunlight reflected by Earth's surface, which causes the greenhouse effect
  - Real surfaces can emit more in certain directions
  - Real surfaces also never have the same overall emission power as a blackbody
- We can define the emissivity: $\varepsilon(T) = \dfrac{E(T)}{E_b(T)}$
  - This is integrated over all directions and wavelengths
  - To simplify calculations, we assume ε is independent of λ (gray surface) and θ (diffuse surface)
  - This gives us the formula we already know: $E(T) \approx \varepsilon\sigma T^4$
- Consider a surface with some incident radiation G; some will be reflected, G_{ref}; some will be absorbed, G_{abs}; some will be transmitted, G_{tran}
  - Define the absorptivity $\alpha = \dfrac{G_{abs}}{G}$, the reflectivity $\rho = \dfrac{G_{ref}}{g}$, and the transmittivity $\tau = \dfrac{G_{tran}}{G}$
  - For a general material these have to sum to 1; for an opaque material $\tau = 0$, so $\alpha + \rho = 1$
  - A blackbody has $\rho = 0$ and so $\alpha = 1$
  - We assume a gray body, where α, ρ, τ are independent of λ, and diffuse, where α, ρ, τ are independent of θ

Lecture 34, Dec 5, 2022

Kirchhoff's Law

- At a specific temperature, $\varepsilon = \alpha$ (emissivity equals absorptivity)
  - This works if both the source of the radiation and the absorber are at the same temperature

- Consider a large enclosure and a small body inside with surface area A, emissivity ε and absorptivity α, both at temperature T
  - Assume the large isothermal cavity acts as a blackbody
  - At thermal equilibrium T is the same for both the small body and enclosure
  - The radiation on the small body per unit area is $G = \sigma T^4$ so $E_{abs} = \alpha G A = \alpha\sigma T^4 A$
  - The emitted radiation is $E_{emit} = \varepsilon\sigma T^4 A$
  - At equilibrium these must be equal, so $E_{abs} = E_{emit}$ and therefore $\varepsilon = \alpha$
- Note this is only true if temperatures for both radiation sources are the same
  - We can use this assumption if the temperatures are similar but not quite equal
  - e.g. 350K vs. 300K is okay for this assumption, but for solar radiation of 5000K vs 300K for a room temperature object this assumption would not apply

View Factors

- The amount of radiation incident on a surface depends on orientation
- Consider two surfaces, i and j, then F_{ij} or $F_{i\to j}$ is the fraction of radiation leaving the surface i that reaches j directly; this is known as a *view factor*
  - Radiation that reaches the other surface via one or more reflections is not counted
  - Note F_{ij} would be from i to j and F_{ji} is from j to i, and they may not be equal
    - Consider the case of surface 1 completely enclosed by 2; F_{11} would be 0 and F_{12} would be 1 (since the surface is completely enclosed), but F_{21} is not necessarily 1, because F_{22} is nonzero
  - F_{11} would be the fraction of energy leaving 1 that reaches 1
    - In the case of a flat or convex surface this would clearly be 0, but if we have a concave surface, this can be nonzero
- The calculation of view factors is done by integration over the shape
  - View factors can be found in tables
  - F_{ij} are functions of dimensions, distance, and orientation
  - Values are tabulated for 3D and 2D (infinitely going into screen) cases

Analyzing Radiation Heat Transfer

- Consider an enclosure with N surfaces
  - Since the region is completely enclosed $\sum_{j=1}^{N} F_{ij} = 1$, i.e. all radiation leaving i must hit a surface in the enclosure (summation rule)
- If the problem is not enclosed, we can make an imaginary enclosure by creating a surface out of an opening, containing $\alpha, \varepsilon, \rho$ of the opening
  - We can usually assume that it's absorbing everything, with $T = 0$ and $\alpha = 1$
- For every pair of surfaces we have a view factor, so in total we have a matrix of view factors where $M_{ij} = F_{ij}$
- Note not all F_{ij} are independent; the actual number of independent view factors is $\frac{N(N-1)}{2}$, based on the summation rule
- Reciprocity rule: $A_i F_{ij} = A_j F_{ji}$ for any pair of surfaces
  - Consider 2 surfaces 1 and 2, which are blackbodies at $T_1 = T_2$ and $\alpha = \varepsilon = 1$
  - Energy leaving 1 and reaching 2 is $E_{b1}(T_1)A_1F_{12}$; energy leaving 2 and reaching 1 is $E_{b2}(T_2)A_2F_{21}$
  - The net energy exchange is $\dot{Q}_{12} = E_{b1}(T_1)A_1F_{12} - E_{b2}(T_2)A_2F_{21}$
  - If we have the same temperature and thus thermal equilibrium, then $\dot{Q}_{12} = 0$ and $E_{b1} = E_{b2}$
  - This gives us $A_1F_{12} = A_2F_{21}$ for a special case, but all of these are geometric parameters independent of T and ε, so this is true in general
- This means that when two areas are equal, the view factors in both directions are equal
- Superposition: We can break up a surface, and its view factor will be the sum of the view factors of the pieces
- Symmetry: if we have an axis of symmetry then view factors are symmetrical

- Example: Imagine a small sphere with a concentric hemisphere, with $A_2 = 2A_1$; find F_{12} and F_{21}
  - Create a third imaginary surface that closes the hemisphere, call it surface 3
  - We know F_{11} is 0 since it's convex
  - Using the plane of symmetry $F_{12} = F_{13} = 0.5$
  - Using reciprocity $F_{21} = \dfrac{A_1}{A_2}F_{12}$ so $F_{21} = 0.25$

Summary

To find the view factors in a system, use the 3 rules:

1. Summation rule: $\sum\limits_{j=1}^{N} F_{ij} = 1$ (sum of all outgoing view factors from a surface is 1)
2. Reciprocity rule: $A_iF_{ij} = A_jF_{ji}$ for any pair of surfaces
3. F_{ii} is 0 for any convex surface

Lecture 35, Dec 6, 2022

Radiation Analysis

- Consider 2 surfaces with A_1, T_1 and A_2, T_2, both blackbodies
  - Energy from 1 to 2 is $A_1E_{b1}(T_1)F_{12}$; energy from 2 to 1 is $A_2E_{b2}(T_2)F_{21}$
  - The net radiative exchange would be $\dot{Q}_{12} = A_1E_{b1}(T_1)F_{12} - A_2E_{b2}(T_2)F_{21}$
  - Using reciprocity $A_1F_{12} = A_2F_{21}$ so $Q_{12} = A_1F_{12}(E_{b1} - E_{b2}) = A_1F_{12}\sigma(T_1^4 - T_2^4)$
- Note for a small body in an enclosure $F_{12} = 1$ which makes our net radiative heat transfer $\dot{Q}_{12} = A_1\sigma(T_1^4 - T_2^4)$
- For a more realistic analysis, assume an isothermal, opaque, diffuse (ε independent of direction), and gray (ε independent of λ) surface
  - Together this gives us ε constant for a material, which is a fair assumption over a small range
  - J is the radiosity, the total radiative energy that leaves a surface per unit area per unit time
  - When incident radiation G hits the surface we have ρG being reflected, αG being absorbed and εE_b being radiated back
  - $J = \rho G + \varepsilon E_b = \varepsilon E_b + (1 - \varepsilon)G \implies G = \dfrac{J - \varepsilon E_b}{1 - \varepsilon}$
  - The net energy leaving the surface per unit area is $\dfrac{\dot{Q}}{A} = J - G \implies \dot{Q} = A(J - G) = A\left(J - \dfrac{J - \varepsilon E_b}{1 - \varepsilon}\right)$
  - $\dot{Q} = A\dfrac{\varepsilon}{1 - \varepsilon}(E_b - J) = \dfrac{E_b - J}{\frac{1-\varepsilon}{\varepsilon A}}$
  - We can think of $E_b - J$ as the driving force of radiative heat exchange, using a resistive approach $R = \dfrac{1 - \varepsilon}{\varepsilon A}$, known as the surface resistance
    - Note if we had a blackbody then $R = 0$ and so $E_b = J$ for a blackbody
- Consider 2 gray surfaces i and j with radiosities J_i, J_j
  - The radiation from i that reaches j is $J_iA_iF_{ij}$; from j to i is $J_jA_jF_{ji}$
  - $\dot{Q}_{ij} = J_iA_iF_{ij} - J_jA_jF_{ji} = A_iF_{ij}(J_i - J_j)$ by reciprocity
  - $\dot{Q}_{ij} = \dfrac{J_i - J_j}{(A_iF_{ij})^{-1}}$
    - $\dfrac{1}{A_iF_{ij}}$ is known as the space resistance
  - The total resistance combines the surface resistance for i, $\dfrac{1 - \varepsilon_i}{A_i\varepsilon_i}$, the space resistance $\dfrac{1}{A_iF_{ij}}$, and

the surface resistance for j, $\dfrac{1-\varepsilon_j}{A_j\varepsilon_j}$, with the driving force being blackbody radiation on both sides

- $\dot{Q}_{ij} = \dfrac{E_{bi}-E_{bj}}{R_{tot}} = \dfrac{\sigma(T_i^4-T_j^4)}{\frac{1-\varepsilon_i}{A_i\varepsilon_i}+\frac{1}{A_iF_{ij}}+\frac{1-\varepsilon_j}{A_j\varepsilon_j}}$
- Example: Consider 2 large parallel plates, $T_1 = 1000\text{K}, \varepsilon_1 = 1, T_2 = 500\text{K}, \varepsilon_2 = 0.8$ with equal area; what is $\dfrac{\dot{Q}_{12}}{A}$?
    - $F_{12} = F_{21} = 1$
    - $\dot{Q}_{12} = \dfrac{\sigma(T_1^4-T_2^4)}{\frac{1-\varepsilon_1}{A\varepsilon_1}+1+\frac{1-\varepsilon_2}{A\varepsilon_2}}$
    - $\dfrac{\dot{Q}_{12}}{A} = \dfrac{\sigma(T_1^4-T_2^4)}{0+1+\frac{1-\varepsilon_2}{\varepsilon_2}} = \varepsilon_2\sigma(T_1^4-T_2^4)$
    - Plugging in values we get 45.5kW/m^2
- Example: An finite system with a groove at 40 degrees, $1000\text{K}, \varepsilon = 0.6, 10\text{mm}$ in the middle; what is the radiation heat flux leaving the groove?
    - Isolate the groove, and create an imaginary surface at the top to enclose the surface; this surface would have 0K and $\varepsilon = 1$ (since $\alpha = 1$); call this surface 2
    - We want $\dot{Q}_{12}$ which would be the amount of radiation escaping the groove
    - Start again with $\dot{Q}_{12} = \dfrac{\sigma(T_1^4-T_2^4)}{\frac{1-\varepsilon_1}{A_1\varepsilon_1}+\frac{1}{A_1F_{12}}+\frac{1-\varepsilon_2}{A_2\varepsilon_2}} = \dfrac{\sigma T_1^4}{\frac{1-\varepsilon_1}{A_1\varepsilon_1}+\frac{1}{A_1F_{12}}}$
    - To find F_{12} we can use reciprocity, and we know $F_{21} = 1$ which gives us $F_{12} = \dfrac{A_2}{A_1} = \sin(20°)$ using geometry
    - Therefore $\dfrac{\dot{Q}_{12}}{A_2} = \dfrac{\sigma T_1^4}{\frac{1-\varepsilon_1}{\varepsilon_1}\frac{A_2}{A_1}+1}$
    - Plugging in values we get 46.2kW/m^2
- If we have multiple surfaces, we still have a single surface resistance, but we have multiple space resistances for multiple sources
    - $\dot{Q}_1 = \displaystyle\sum_{i=1}^{N}\dot{Q}_{1i}$
    - Due to conservation of energy $\dfrac{E_{b1}-J_1}{R_1} = \displaystyle\sum_{i=1}^{N}\dfrac{J_1-J_i}{R_{1i}}$
- Note even when $F_{11} > 0$, we still have $\dot{Q}_{11} = 0$ since we assume an isothermal surface
- In a system with multiple surfaces we get a system of resistances, which we can solve by assessing each node and noting that the heat in equals the heat out for all intermediate nodes

Summary

The total radiative heat transfer between two surfaces i and j is given by

$$\dot{Q}_{ij} = \frac{E_{bi}-E_{bj}}{R_{tot}} = \frac{\sigma(T_i^4-T_j^4)}{\frac{1-\varepsilon_i}{A_i\varepsilon_i}+\frac{1}{A_iF_{ij}}+\frac{1-\varepsilon_j}{A_j\varepsilon_j}}$$

in which $\dfrac{1-\varepsilon}{A\varepsilon}$ terms are the surface resistances, $\dfrac{1}{A_iF_{ij}}$ is the space resistance, and the driving force is the difference in blackbody radiation between the surfaces; for a system with multiple surfaces, each surface has its own surface resistance, and each pair of surfaces has a space resistance between them