Lecture 35, Dec 2, 2022

Models of the Fluid

- To develop any governing equation, we start from physical principles (e.g. conservation of mass, energy), use a suitable model of the fluid, and then derive mathematical equations
  - We have seen the finite control volume and finite system, now we will see the infinitesimal control volume and infinitesimal system
- We take an infinitesimally small control volume and system moving with the flow, such that all properties in this volume are constant

Substantial Derivative

- Consider an infinitesimally small fluid element, at point 1 at $t = t_1$ and travelling to point 2 at $t = t_2$
- Let $\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ be the velocity field, where u, v, w are functions of (x, y, z, t)
  - Let $\rho = \rho(x, y, z, t)$ be the density
- Initially the fluid particle has $\rho_1 = \rho(x_1, y_1, z_1, t_1)$ and velocity $\vec{V}_1$; at time $t = t_2$ the particle has $\rho_2 = \rho(x_2, y_2, z_2, t_2)$ and $\vec{V}_2$
- Taylor expand density around point 1: $\rho_2 = \rho_1 + \frac{\partial \rho}{\partial x}(x_2 - x_1) + \frac{\partial \rho}{\partial y}(y_2 - y_1) + \frac{\partial \rho}{\partial z}(z_2 - z_1) + \frac{\partial \rho}{\partial t}(t_2 - t_1) + \cdots$
  - $\frac{\rho_2 - \rho_1}{t_2 - t_1} = \frac{\partial \rho}{\partial x}\frac{x_2 - x_1}{t_2 - t_1} + \frac{\partial \rho}{\partial y}\frac{y_2 - y_1}{t_2 - t_1} + \frac{\partial \rho}{\partial z}\frac{z_2 - z_1}{t_2 - t_1} + \frac{\partial \rho}{\partial t}$
  - If we take $\lim_{t_2 \to t_1}$, we get $\frac{\mathrm{D}\rho}{\mathrm{D}t} = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t}$
    - The notation $\frac{\mathrm{D}\rho}{\mathrm{D}t}$ indicates that we're following the same system
    - This is called a *substantial derivative*
- The substantial derivative $\frac{\mathrm{D}}{\mathrm{D}t}$ is Lagrangian, while the right hand side with all the partial derivatives is Eulerian

Definition

The substantial derivative operator is defined as

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla}$$

where $\vec{V} \cdot \vec{\nabla}$ is the convective derivative

- The substantial derivative is made of the local derivative, time rate of change at a fixed point due to unsteady fluctuations, and the *convective derivative* $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$, which is the time rate of change as a result of the movement of fluid
- The substantial derivative is a total derivative with respect to time

Divergence of Velocity

- $\vec{\nabla} \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ can be interpreted as the time rate of change of the volume of an infinitesimal moving system per unit volume
- As the system is moving, its volume is continuously changing
- $\vec{\nabla} \cdot \vec{V} = \frac{1}{\delta\Omega}\frac{\mathrm{D}(\delta\Omega)}{\mathrm{D}t}$
- Note that if we have an incompressible flow, constant density means $\delta\Omega_{sys}$ must be constant, so $\frac{\mathrm{D}(\delta\Omega)}{\mathrm{D}t} = 0$, this means $\vec{\nabla} \cdot \vec{V} = 0$

General Continuity Equation

- We're starting with mass conservation, move through the 4 different models to get the continuity equation
- Recall for the finite control volume we have $\frac{\mathrm{d}}{\mathrm{d}t}\iiint_\Omega \rho\,\mathrm{d}\Omega + \oiint_S \rho\vec{V}\cdot\mathrm{d}\vec{S} = 0$ which we derived from RTT
- For the finite system, we follow the system and take $\frac{\mathrm{D}m_{sys}}{\mathrm{D}t}$, which by mass conservation should be 0, which gives us $\frac{\mathrm{D}}{\mathrm{D}t}\iiint_\Omega \rho\,\mathrm{d}\Omega = 0$
- For the infinitesimal control volume:
  - Along a side we have the flow rate as $\left(\rho u + \frac{\partial(\rho u)}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y\,\mathrm{d}z$
  - The net mass flow rate out of the control volume is $\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho u)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
  - This is equal to $-\frac{\partial}{\partial t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ due to conservation of mass
  - Combined this gives us $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0$
- Now consider the infinitesimal system:
  - $\frac{\mathrm{D}(\delta m)}{\mathrm{D}t} = 0$ where $\delta m = \rho\delta\Omega$
  - $\frac{\mathrm{D}(\delta m)}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}(\rho\delta\Omega) = \rho\frac{\mathrm{D}}{\mathrm{D}t}(\Delta\Omega) + \delta\Omega\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$
  - $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\left(\frac{1}{\delta\Omega}\frac{\mathrm{D}(\Delta\Omega)}{\mathrm{D}t}\right) = 0$
  - This gives us $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\vec{\nabla}\cdot\vec{V} = 0$
- This gave us 4 general forms of the continuity equation:
  1. $\frac{\mathrm{d}}{\mathrm{d}t}\iiint_\Omega \rho\,\mathrm{d}\Omega + \oiint_S \rho\vec{V}\cdot\mathrm{d}\vec{S} = 0$
  2. $\frac{\mathrm{D}}{\mathrm{D}t}\iiint_\Omega \rho\,\mathrm{d}\Omega = 0$
  3. $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0$
  4. $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\vec{\nabla}\cdot\vec{V} = 0$
- For a steady flow, $\frac{\partial}{\partial t} = 0$ but $\frac{\mathrm{D}}{\mathrm{D}t} \neq 0$, so we have:
  1. $\oiint_S \rho\vec{V}\cdot\,\mathrm{d}\vec{S} = 0$
  2. $\frac{\mathrm{D}}{\mathrm{D}t}\iiint_\Omega \rho\,\mathrm{d}\Omega = 0$
  3. $\vec{\nabla}\cdot(\rho\vec{V}) = 0$
  4. $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\vec{\nabla}\cdot\vec{V} = 0$
- For an compressible flow, ρ is constant and so:
  1. $\oiint_S \rho\vec{V}\cdot\mathrm{d}\vec{S} = 0$
  2. $\frac{\mathrm{D}}{\mathrm{D}t}\Omega = 0$
  3. $\vec{\nabla}\cdot\vec{V} = 0$
     - Note 3 and 4 became the same
- All these forms are mathematically equivalent