Lecture 32/33, Nov 25, 2022

Bernoulli's Equation

- Under the assumption of incompressible flow (constant ρ) we can integrate Euler's equation and obtain Bernoulli's equation
- $\frac{V^2}{2} + gz + \frac{p}{\rho} = \text{const}$
- Bernoulli's equation is an energy conservation equation; all terms have units of energy per unit mass
  - $\frac{V^2}{2}$ is kinetic energy per unit mass
  - gz is potential energy per unit mass
  - $\frac{p}{\rho}$ is pressure energy per unit mass

Summary

For steady, inviscid flow along a streamline, Euler's equation is given by

$$V\,\mathrm{d}V + g\,\mathrm{d}z + \frac{1}{\rho}\,\mathrm{d}p = 0$$

If the flow is also incompressible, then Bernoulli's equation applies:

$$\frac{V^2}{2} + gz + \frac{p}{\rho} = \text{const}$$

- Note that Euler's/Bernoulli's equation only works along a streamline!
- Additionally since they rely on energy conservation, between the two points there must be no heat loss and no shaft work
- If we apply the same analysis normal to the streamline we get $\frac{\mathrm{d}p}{\rho} + \frac{V^2}{R}\,\mathrm{d}n + g\,\mathrm{d}z = 0$
  - For a straight streamline $R \to \infty$ and we just have $\frac{\mathrm{d}p}{\rho} + g\,\mathrm{d}z = 0$
  - If the flow is incompressible we can integrate this and get $p_1 - p_2 = \rho g(z_2 - z_1)$, which is the hydrostatic equation, but this time for a straight, incompressible flow

Static, Dynamic, Total and Stagnation Pressures

- Multiplying the Bernoulli equation by ρ we get $p + \rho\frac{V^2}{2} + \rho gz = P_T$
  - p is the static pressure, the pressure in the flow that does not incorporate any dynamic effects
  - $\rho\frac{V^2}{2}$ is the dynamic pressure, the pressure rise when the fluid is stopped isentropically
  - ρgz is the hydrostatic pressure (but not exactly, since it depends on the reference level for z)
  - P_T is the total pressure
  - $p + \rho\frac{V^2}{2}$ is the stagnation pressure, the pressure observed when the fluid is brought to a stop (includes both the static and dynamic pressures)
- Static and stagnation pressures can be measured through a piezometer tube and a pitot tube, from which the flow velocity can be calculated

Reynolds Transport Theorem

- Two approaches of examining the flow:
  - Control volume approach (Eulerian): consider a region fixed in space, which fluid can flow in or out of

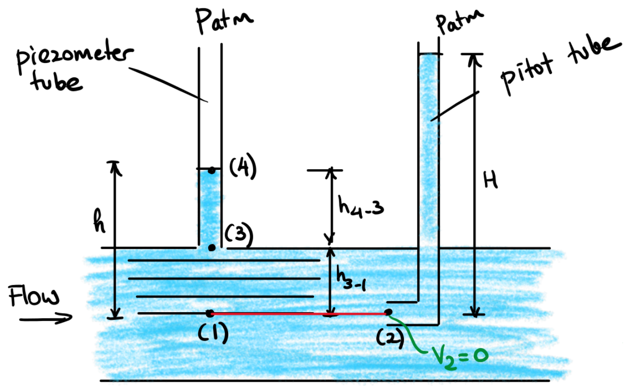

Figure 1: Measurement of static and dynamic pressures

  - System approach (Lagrangian): consider a fixed collection of fluid particles, which moves with the flow
- Equations used in solid analysis (e.g. Newton's laws) apply to systems, but for fluids it's easier to use control volumes; Reynolds Transport Theorem links the two approaches
- Let B be some mass dependent property and $B = mb$ where b is that property per unit mass
  - $B_{sys} = \int_{sys} b\rho \mathrm{d}V \implies \frac{\mathrm{d}B_{sys}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{sys} b\rho \mathrm{d}V$
  - $B_{cv} = \int_{cv} b\rho \mathrm{d}V \implies \frac{\mathrm{d}B_{cv}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{cv} b\rho \mathrm{d}V$

Equation

Reynolds Transport Theorem:

$$\frac{\mathrm{d}B_{sys}}{\mathrm{d}t} = \frac{\mathrm{d}B_{cv}}{\mathrm{d}t} + \oiint_S \rho b\vec{V}\cdot \mathrm{d}\vec{A}$$

where $\mathrm{d}\vec{A} = \vec{n}\mathrm{d}A$ and S is the boundary of the control volume

- In the simplified case of single-inlet, single-outlet, 1D flow along a streamtube, $\frac{\mathrm{d}B_{sys}}{\mathrm{d}t} = \frac{\mathrm{d}B_{cv}}{\mathrm{d}t} + \dot{B}_{out} - \dot{B}_{in} = \frac{\mathrm{d}B_{cv}}{\mathrm{d}t} + \dot{m}_{out}b_{out} - \dot{m}_{in}b_{in}$
- Using this we can derive the most general form of the continuity equation as $\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho \,\mathrm{d}V + \oiint_S \rho\vec{V}\cdot \mathrm{d}\vec{A} = 0$
- Note the selection of control volume can make a problem easier; e.g. in the case of a body moving through a fluid, it's often best to have the control volume move with the body, so flow inside the control volume is steady