Lecture 30, Nov 18, 2022

Fluids in Rotational Rigid-Body Motion

- Consider a cylindrical container with radius R partially filled with liquid, rotated about its centre axis at a constant velocity ω
- Assume no shear, what is the free surface $z = z_s$?
- Using a cylindrical coordinate system (r, θ, z), we have $\begin{cases} a_r = -r\omega \\ a_\theta = 0 \\ a_z = 0 \end{cases}$
  - a_r comes from centripetal acceleration
- Using the hydrostatic equation again, $-\left(\frac{\partial p}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\vec{e}_\theta + \frac{\partial p}{\partial z}\vec{e}_z\right) - \rho g\vec{e}_z = \rho(-r\omega^2)\vec{e}_r$
  - Notice the gradient in cylindrical coordinates has a $\frac{1}{r}$ in the θ term
- Matching the terms: $\begin{cases} -\frac{\partial p}{\partial r} = -\rho r\omega^2 \\ -\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \\ -\frac{\partial p}{\partial z} = \rho g \end{cases} \implies \mathrm{d}p = \frac{\partial p}{\partial r}\,\mathrm{d}r + \frac{\partial p}{\partial z}\,\mathrm{d}z = \rho r\omega^2\,\mathrm{d}r - \rho g\,\mathrm{d}z$
- Integrating this we have $p = \frac{\rho r^2\omega^2}{2} - \rho g z + C$
- Again using the boundary condition that $p = p_{atm}$ at $z = z_s$ we have $p_{atm} = \frac{\rho r^2\omega^2}{2} - \rho g z_s + C \implies z_s = \frac{\omega^2}{2g}r^2 + \frac{C - p_{atm}}{\rho g} = \frac{\omega^2}{2g}r^2 + C_1$
- Therefore the free surface is a paraboloid
- Again from the fact that at rest the fluid surface is flat and the volume stays the same, integrating the volume of the paraboloid we can solve for $C_1 = H - \frac{\omega^2 R^2}{4g} \implies z_s = \frac{\omega^2}{4g}(2r^2 - R^2) + H$

Summary

The free surface for a fluid undergoing rotational rigid-body motion is a paraboloid given by

$$z_s(r) = \frac{\omega^2}{4g}(2r^2 - R^2) + H$$

with pressure given by

$$p = \frac{\rho r^2\omega^2}{2} - \rho g z + C$$

Fluid in Motion (Overview)

- Lagrangian vs. Eulerian descriptions:
  - The Lagrangian method describes the motion of individual particles over time, modelling using Newton's laws
    - $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + k(t)\hat{k}$
    - To describe the whole fluid we need to do this for a large number of particles
  - The Eulerian method instead considers regions of space and properties such as the overall fluid velocity in a certain region
    - $\vec{V} = \vec{V}(x, y, z, t)$
  - Eulerian method is a lot easier to work with
- 3 concepts are used to visualize a flow:

- Streamlines: Imaginary lines that are tangent to the local fluid velocity at every point at an instant in time
  - Streamlines can vary through time
  - By definition, fluid cannot ever flow across a streamline
  - Streamlines can be determined using Particle Image Velocimetry (PIV)
  - Streamtubes: A set of streamlines passing through all points of a closed curve
    - Fluid can't cross a streamtube, so these can be used to isolate a region of a flow
  - Stream filament: A streamtube that is thin enough that the cross-sectional velocity is effectively constant
- Pathlines: The line traced out by a specific fluid particle as it moves in time
  - Lagrangian concept
  - Experimentally, they can be visualized by putting tracer particles in the fluid and taking long-exposure photographs
- Streaklines: The line formed by connecting all fluid particles that pass through a point
  - Experimentally, they can be visualized by injecting dye or smoke into the fluid at a particular fixed point

* Steady flows are flows in which fluid properties at a point do not change with time
  - Fluid properties are a function of space only, not time
  - In a steady flow, the streamline, streakline, and pathline all coincide
* Unsteady flows are flows in which the properties can be time-dependent
  - In an unsteady flow, the streamline, streakline and pathline are different
  - Two particles that pass through the same point don't necessarily end up in the same place, so the streakline connecting the points won't be the same as the pathline
  - The streamline will vary with time